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ON FINDING QUADRATIC FACTORS
OF POLYNOMIALS

A Thesis

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ABSTRACT

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It is frequently necessary in Engineering Problems to determine the roots of an algebraic equation. Few simple methods exist which permit the rapid determination, by hand calculation of complex zeros of polynomials. One of these methods has been proposed by R. Oldenburger and consists in the repetition of a right to left synthetic division, the same method being also available for determining real zeros. However this method does not converge for every polynomial.

The main purpose of this work has been the study of the convergence of Oldenburger's method which is considered as an iteration process. The conditions of convergence are determined. A geometrical interpretation is given in the complex plane for the particular case of lower order polynomials.

The convergence occurs according to certain patterns, depending upon the roots of the given equation, and it is shown how the knowledge of these patterns allows the finding of the roots to be accelerated.

Finally a general study of diverging cases is made and it is shown that the method can still be used.

AUTHOR

INTRODUCTION

In automatic controls and other branches of engineering the problem of finding roots of algebraic equations is a very important one, especially when, as frequently occurs in engineering, problems of stability or eigenvalue determination are involved. When those equations are of second, third or fourth order, formulas in closed form are available to determine the roots. If one has access to a digital computer and if the necessary subroutines are available, the zeros of algebraic functions may be found without too much difficulty whatever the order of the equation might be. However, when an equation of fourth degree or higher is encountered in an everyday problem, it is often useful to solve it by some rapid method of hand calculation.

Many methods have been presented for finding roots of algebraic equations, most of which apply for finding real roots. Some of these methods are synthetic division, Newton's method and Horner's method [1]. The choice is more restricted, however, when it comes to complex roots or when two or more real roots are identical or almost identical. The classical method of Graeffe [1], [2] is still one of the most popular although it may become very cumbersome and impractical to use for fast hand calculation especially when

some of the roots of the studied equation are close in magnitude. Other processes such as those of Bairstow, Hitchcock [3], [4] and Lin [5], [6], [7] are almost the only ones of some practical use. The respective properties of each of these methods, from the point of view of fast hand calculation have been considered in more detail by Moore in an earlier work. However, these methods do not converge to a solution in every case.

In 1958, Oldenburger [8] presented a new method for determining roots of polynomials. An algorithm was defined which introduced an iterative procedure allowing factorization of a given polynomial into two polynomials, one of which was a linear or a quadratic factor. By repetition of this algorithm on the remaining polynomial, it is possible to factor the original polynomial into linear and quadratic factors. Although the method is especially meant for hand and slide rule calculation it can also be very easily programmed on a digital computer or used with a desk calculator.

Although the philosophy of Oldenburger's method is somewhat similar to that of Lin, it has the advantage of requiring a lesser number of steps for each iteration and of being very usable for hand and slide rule calculation. Furthermore, it will converge in most of the cases in which the other methods will not. For instance, Oldenburger's method generally converges towards the root of larger magnitude, whereas Lin's method converges towards the root of

smaller magnitude and usually fails to converge towards the larger roots. Similarly Oldenburger's method may fail to converge in some cases or the convergence might be very slow or difficult to see. In a recent work, Moore and Oldenburger [9] studied the convergence of the iteration when applied to find real zeros of algebraic equations. They gave several transformations which modify an equation in which the iteration diverges so that the method will converge when applied to the transformed equation.

However, the existence of convergence or divergence has not been studied for Oldenburger's method when a quadratic factor is sought although this is the case of great interest because it allows complex roots to be found.

It is the purpose of this work to determine from a theoretical viewpoint the reasons for which the iteration will converge or diverge when the method proposed by Oldenburger is applied to find a quadratic factor. A criterion for asymptotic stability of the iteration process is presented, and it is compared to that found when Oldenburger's method is applied for the search of linear factors.

To make this study from a theoretical viewpoint it is assumed that the values found at the various stages of the iteration are close enough to the searched values so that a linearization of the equations could be made. The first order terms in the difference between the exact and approximate value are the only ones considered although, as

numerous examples illustrate, this condition does not have to be satisfied actually. In order to handle the calculations in an easier way, the relations between two successive values of the iteration are written in matrix form.

An analysis of the conditions for convergence obtained for the quadratic factor search shows the different ways in which the convergence of the iteration proceeds to this factor depending first upon the nature of the roots of this quadratic factor and also upon the characteristic roots of the matrix which defines the iteration process. A few examples are given.

Finally the eventual user of the method is shown how to interpret the results of the iteration in order to save many calculations when convergence is slow. Further, in some cases for which the iteration procedure diverges the quadratic factor can often be found by a method of "regula falsi".

1. THEORETICAL STUDY OF THE CONVERGENCE
OF OLDENBURGER'S METHOD FOR QUADRATIC FACTORS

Consider a polynomial $P_n(z)$ given by

$$P_n(z) = \sum_{k=0}^n a_k z^k \quad (1)$$

where the variable z may be complex.

The coefficients a_k ($k=0, 1, 2, \dots, n$) of the polynomial are real. It is furthermore assumed without loss of generality that $a_n = 1$.

Oldenburger's method for finding quadratic factors of $P_n(z)$ is explained in detail in references [8], [9]. It is a process by which the polynomial $P_n(z)$ is divided by a trial quadratic factor $p(z)$ given by

$$p(z) = \beta + \alpha z + z^2 \quad (2)$$

The division is carried out in a right to left manner until all coefficients a_k of $P_n(z)$ are exhausted with the exception of the first one which is kept in order to form the next trial factor.

This right to left division can be considered as a division of $P_n(z)$ by $p(z)$ along increasing powers of z and can be represented as

$$P_n(z) = (\beta + \alpha z + z^2) \sum_{k=0}^{n-2} \frac{b_k}{\beta} z^k + r_{n-1} z^{n-1} + r_n z^n \quad (3)$$

The coefficients b_k of the quotient are defined as functions of α and β by the recurrence formula

$$\begin{aligned} a_k &= b_k + \frac{\alpha}{\beta} b_{k-1} + \frac{1}{\beta} b_{k-2} \\ k &= 0, 1, 2, \dots, n-1 \end{aligned} \quad (4)$$

where b_{-1} and b_{-2} are each zero.

The coefficients of the remainder terms are given by

$$\begin{aligned} r_{n-1} &= b_{n-1} \\ r_n &= a_n - \frac{b_{n-2}}{\beta} \end{aligned} \quad (5)$$

For the purpose of comparison Oldenburger's disposition of the calculations is shown in Figure 1 where the b_k 's are given by Equation (4) and the c_k 's by

$$c_k = a_{k+1} - \frac{b_{k-1}}{\beta} \quad (6)$$

The next trial factor $p_1(z)$ as defined by Oldenburger's method is the last remainder which yields a second order factor in the array of Figure 1, i.e.

$$p_1(z) = \beta_1 + \alpha_1 z + z^2 \quad (7)$$

where

$$\begin{array}{cccccccc}
 a_n & a_{n-1} & a_{n-2} & \dots & a_k & \dots & a_3 & a_2 & a_1 & a_0 & \left| \begin{array}{c} 1 \quad \alpha \quad \beta \\ a_0 \quad b_1 \quad \dots \quad b_{n-3} \quad b_{n-2} \\ \beta \quad \beta \quad \beta \quad \beta \end{array} \right.
 \end{array}$$

$$\begin{array}{c}
 \frac{a_0}{\beta} \quad \frac{\alpha a_0}{\beta} \quad \frac{b_1}{c_1} \\
 \frac{b_1}{\beta} \quad \frac{\alpha b_1}{\beta} \quad \frac{b_2}{c_2}
 \end{array}$$

$$\begin{array}{c}
 \frac{b_{k-2}}{\beta} \quad \frac{c_{k-1}}{\beta} \quad \frac{\alpha b_{k-1}}{\beta} \\
 \frac{b_k}{\beta} \quad \frac{c_k}{\beta} \quad \frac{\alpha b_{k-1}}{\beta}
 \end{array}$$

$$\begin{array}{c}
 \frac{b_{n-2}}{\beta} \quad \frac{c_{n-1}}{\beta} \quad \frac{\alpha b_{n-2}}{\beta} \\
 \frac{b_{n-1}}{\beta} \quad \frac{c_{n-2}}{\beta} \quad \frac{\alpha b_{n-1}}{\beta}
 \end{array}$$

$$\begin{array}{c}
 b_k = c_{k-1} - \frac{\alpha b_{k-1}}{\beta} \\
 c_k = a_{k+1} - \frac{b_{k-1}}{\beta} \\
 a_k = b_k + \frac{\alpha b_{k-1}}{\beta} + \frac{b_{k-2}}{\beta}
 \end{array}$$

Figure 1

Oldenburger's Right to Left Synthetic Division for

Determination of Quadratic Factors

$$\beta_1 = b_{n-2} \quad (8)$$

and

$$\alpha_1 = c_{n-2} \quad (9)$$

It is shown in Appendix A that the general b_k term can be written as a determinant of the form

$$b_k = (-1)^k \begin{vmatrix} a_0 & a_1 & a_2 & \dots & \dots & a_{k-2} & a_{k-1} & a_k \\ 1 & \frac{\alpha}{\beta} & \frac{1}{\beta} & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & \frac{\alpha}{\beta} & \frac{1}{\beta} & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \dots\dots\dots & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \dots\dots\dots & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & 1 & \frac{\alpha}{\beta} & \frac{1}{\beta} & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \frac{\alpha}{\beta} & \frac{1}{\beta} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \frac{\alpha}{\beta} \end{vmatrix} \quad (10)$$

Note that the first row is composed of the coefficients of the $(k + 1)$ lower order terms of the polynomial, the main diagonal of $\frac{\alpha}{\beta}$ terms, and the two diagonals on each side of it of 1's and $\frac{1}{\beta}$ terms respectively. All other terms are equal to zero.

Combining Equations (8), (9) and (10), the coefficients of the next trial quadratic factor are given by the

determinants

$$\alpha_1 = (-1)^{n-2} \begin{vmatrix} a_0 & a_1 & a_2 & \dots & a_{n-4} & a_{n-3} & a_{n-1} \\ 1 & \frac{\alpha}{\beta} & \frac{1}{\beta} & \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & \frac{\alpha}{\beta} & \frac{1}{\beta} & \cdot & \cdot & \cdot \\ \cdot & \cdot & \dots & \dots & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 & \frac{\alpha}{\beta} & \frac{1}{\beta} & \cdot \\ \cdot & \cdot & \cdot & \cdot & 1 & \frac{\alpha}{\beta} & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \frac{1}{\beta} \end{vmatrix} \quad (11)$$

$$\beta_1 = (-1)^{n-2} \begin{vmatrix} a_0 & a_1 & a_2 & \dots & a_{n-4} & a_{n-3} & a_{n-2} \\ 1 & \frac{\alpha}{\beta} & \frac{1}{\beta} & \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & \frac{\alpha}{\beta} & \frac{1}{\beta} & \cdot & \cdot & \cdot \\ \cdot & \cdot & \dots & \dots & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \dots & \dots & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 & \frac{\alpha}{\beta} & \frac{1}{\beta} & \cdot \\ \cdot & \cdot & \cdot & \cdot & 1 & \frac{\alpha}{\beta} & \frac{1}{\beta} \\ \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \frac{\alpha}{\beta} \end{vmatrix} \quad (12)$$

When the trial factor $p(z)$ is an exact factor of $P_n(z)$ the

following notation is used:

$$p^*(z) = B + A z + z^2 \quad (13)$$

$$P_n(z) = (B + A z + z^2) \sum_{k=0}^{n-2} \frac{b_k^*}{B} z^k \quad (14)$$

In other words p^* , A , B , b_k^* are the final values of p , α , β and b_k . Using these values, Equations (4) and (5) become respectively

$$a_k = b_k^* + \frac{A}{B} b_{k-1}^* + \frac{1}{B} b_{k-2}^* \quad (15)$$

$$k = 1, 2, \dots, n$$

where b_{-1}^* , b_{-2}^* , b_{n-1}^* and b_n^* are each zero, and

$$\frac{b_{n-2}^*}{B} = 1 \quad (16)$$

$$\begin{aligned} r_{n-1}^* &= 0 \\ r_n^* &= 0 \end{aligned} \quad (17)$$

and

$$b_k^* = (-1)^k \begin{vmatrix} a_0 & a_1 & a_2 & \dots & \dots & a_{k-2} & a_{k-1} & a_k \\ 1 & \frac{A}{B} & \frac{1}{B} & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & \frac{A}{B} & \frac{1}{B} & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \dots & \dots & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \dots & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & 1 & \frac{A}{B} & \frac{1}{B} & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \frac{A}{B} & \frac{1}{B} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \frac{A}{B} \end{vmatrix} \quad (18)$$

In order to consider the convergence of the iteration process, define $h_{k,u}$ by

$$h_{k,u} = b_k^* - b_{k,u} \quad (19)$$

i.e., $h_{k,u}$ is the difference between the exact value of the coefficient b_k^* and the value found at the u^{th} stage of the iteration. Similarly, define $x_{1,u}$ and $x_{2,u}$ by

$$x_{1,u} = A - \alpha_u \quad (20)$$

$$x_{2,u} = B - \beta_u$$

It can be seen that $x_{1,u}$ and $x_{2,u}$ represent respectively

the differences between the final values A and B and those obtained at the u^{th} stage of the iteration. The vector (x_1, x_2) used frequently in what follows is defined as the "error vector" in the α, β -plane.

Equation (20) may be rewritten:

$$\begin{aligned}\alpha_u &= A \left(1 - \frac{x_{1,u}}{A}\right) \\ \beta_u &= B \left(1 - \frac{x_{2,u}}{B}\right)\end{aligned}\tag{21}$$

It is assumed in the following development that $\frac{x_1}{A}$ and $\frac{x_2}{B}$ are small quantities compared to one so that their higher powers can be neglected. This implies that $p(z)$ is a good approximation to $p^*(z)$. In actual practice, however, this condition need not be rigorously satisfied and convergence may be obtained when $p(z)$ is very different from $p^*(z)$.

Substituting (21) into (4) and taking (20) into consideration yields:

$$\begin{aligned}b_k^* - h_k &= a_k - \frac{A}{B} \left(1 - \frac{x_1}{A}\right) \left(1 + \frac{x_2}{B}\right) (b_{k-1}^* - h_{k-1}) \\ &\quad - \frac{1}{B} \left(1 + \frac{x_2}{B}\right) (b_{k-2}^* - h_{k-2})\end{aligned}\tag{22}$$

Applying (15), one obtains after neglecting the terms of order higher than one in x_1, x_2 the expression

$$\begin{aligned}h_k &= -\frac{A}{B} h_{k-1} - \frac{1}{B} h_{k-2} + \frac{A}{B} \frac{b_{k-1}^*}{B} x_2 + \frac{1}{B} \frac{b_{k-2}^*}{B} x_2 \\ &\quad - \frac{b_{k-1}^*}{B} x_1\end{aligned}\tag{23}$$

In Appendix B it is shown that Equation (23) can be rewritten in the form of Equation (24) where the coefficients of the x_1 and x_2 terms are determinants.

$$\begin{aligned}
 h_k = (-1)^k & \left\{ \begin{array}{ccccc|c} \frac{b^*_0}{B} & \frac{b^*_1}{B} & \dots\dots & \frac{b^*_{k-2}}{B} & \frac{b^*_{k-1}}{B} & \\ 1 & \frac{A}{B} & \frac{1}{B} & \cdot & \cdot & \\ \cdot & \dots\dots\dots & & & \cdot & \\ \cdot & \cdot & 1 & \frac{A}{B} & \frac{1}{B} & \\ \cdot & \cdot & \cdot & 1 & \frac{A}{B} & \end{array} \right. x_1 \\
 & - \left\{ \begin{array}{ccccc|c} \frac{b^*_0}{B} & \frac{b^*_1}{B} & \dots\dots & \frac{b^*_{k-2}}{B} & \frac{b^*_{k-1}}{B} & 0 \\ 1 & \frac{A}{B} & \frac{1}{B} & \cdot & \cdot & \cdot \\ \cdot & \dots\dots\dots & & & \cdot & \cdot \\ \cdot & \cdot & \dots\dots\dots & & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 & \frac{A}{B} & \frac{1}{B} \\ \cdot & \cdot & \cdot & \cdot & 1 & \frac{A}{B} \end{array} \right. x_2
 \end{aligned} \tag{24}$$

From Equations (8), (9), (6), (20) and (24) the relations between the error vectors at the u and the $u+1$ iterations become

$$\begin{aligned}
 x_{1, u+1} &= m_{11} x_{1,u} + m_{12} x_{2,u} \\
 x_{2, u+1} &= m_{21} x_{1,u} + m_{22} x_{2,u}
 \end{aligned}
 \tag{25}$$

where

$$m_{11} = (-1)^n \begin{vmatrix} \frac{b^*_0}{B} & \frac{b^*_1}{B} & \frac{b^*_2}{B} & \dots & \frac{b^*_{n-5}}{B} & \frac{b^*_{n-4}}{B} & 0 \\ 1 & \frac{A}{B} & \frac{1}{B} & \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & \frac{A}{B} & \frac{1}{B} & \cdot & \cdot & \cdot \\ \cdot & \cdot & \dots & \dots & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 & \frac{A}{B} & \frac{1}{B} & \cdot \\ \cdot & \cdot & \cdot & \cdot & 1 & \frac{A}{B} & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & 0 & \frac{1}{B} \end{vmatrix}
 \tag{26}$$

$$m_{12} = (-1)^{n-1} \begin{vmatrix} \frac{b^*_0}{B} & \frac{b^*_1}{B} & \frac{b^*_2}{B} & \dots & \frac{b^*_{n-5}}{B} & \frac{b^*_{n-4}}{B} & \frac{b^*_{n-3}}{B} & 0 \\ 1 & \frac{A}{B} & \frac{1}{B} & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & \frac{A}{B} & \frac{1}{B} & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \dots & \dots & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \dots & \dots & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & 1 & \frac{A}{B} & \frac{1}{B} & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \frac{A}{B} & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 0 & \frac{1}{B} \end{vmatrix}
 \tag{27}$$

$$m_{21} = (-1)^n \begin{vmatrix} \frac{b^*_0}{B} & \frac{b^*_1}{B} & \frac{b^*_2}{B} & \dots & \frac{b^*_{n-5}}{B} & \frac{b^*_{n-4}}{B} & \frac{b^*_{n-3}}{B} \\ 1 & \frac{A}{B} & \frac{1}{B} & \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & \frac{A}{B} & \frac{1}{B} & \cdot & \cdot & \cdot \\ \cdot & \cdot & \dots\dots\dots & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 & \frac{A}{B} & \frac{1}{B} & \cdot \\ \cdot & \cdot & \cdot & \cdot & 1 & \frac{A}{B} & \frac{1}{B} \\ \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \frac{A}{B} \end{vmatrix} \quad (28)$$

$$m_{22} = (-1)^{n-1} \begin{vmatrix} \frac{b^*_0}{B} & \frac{b^*_1}{B} & \frac{b^*_2}{B} & \dots & \frac{b^*_{n-5}}{B} & \frac{b^*_{n-4}}{B} & \frac{b^*_{n-3}}{B} & 0 \\ 1 & \frac{A}{B} & \frac{1}{B} & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & \frac{A}{B} & \frac{1}{B} & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \dots\dots\dots & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \dots\dots\dots & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & 1 & \frac{A}{B} & \frac{1}{B} & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \frac{A}{B} & \frac{1}{B} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \frac{A}{B} \end{vmatrix} \quad (29)$$

Equations (25) can be rewritten in the matrix form

$$\begin{bmatrix} x_{1,u+1} \\ x_{2,u+1} \end{bmatrix} = \begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix} \begin{bmatrix} x_{1,u} \\ x_{2,u} \end{bmatrix} \quad (30)$$

or

$$X_{u+1} = M X_u \quad (31)$$

As can be seen the matrix relating two successive steps of the iteration depends only on the final values A,B of the exact quadratic factor $p^*(z)$ and on the b_k^* 's and is therefore independent of the values taken during the iteration.

Oldenburger has shown [10] that when an equation of the form of (31) relates two successive steps of an iteration, a necessary and sufficient condition of convergence is that the characteristic roots λ_i of the M-matrix lie inside the unit circle, that is:

$$|\lambda_i| < 1 \quad (32)$$

where the values of λ_i are given by the solutions of

$$M - \lambda I = 0 \quad (33)$$

and I is the identity matrix.

Condition (33) is now considered for the particular case of the right to left synthetic division, and the condition of convergence given as a function of the roots of the polynomial and of the quotient of the division of $P_n(z)$ by $p^*(z)$.

Let the zeros of $p^*(z)$ be denoted by z_1 and z_2 , i.e.

$$p^*(z) = (z - z_1)(z - z_2) \quad (34)$$

It is shown in Appendix C that the values of λ_i are given by either of the following equivalent expressions

$$\lambda_i = m_{11} - m_{12} z_i \quad (35)$$

or

$$\lambda_1 = 1 - \frac{1}{z_2^{n-2}} \prod_2(z_2) \quad (36)$$

$$\lambda_2 = 1 - \frac{1}{z_1^{n-2}} \prod_2(z_1)$$

where $\prod_2(z) = \frac{P_n(z)}{p^*(z)}$

or

$$\lambda_1 = 1 - \left(1 - \frac{z_3}{z_2}\right) \left(1 - \frac{z_4}{z_2}\right) \dots \left(1 - \frac{z_n}{z_2}\right) \quad (37)$$

$$\lambda_2 = 1 - \left(1 - \frac{z_3}{z_1}\right) \left(1 - \frac{z_4}{z_1}\right) \dots \left(1 - \frac{z_n}{z_1}\right)$$

The convergence condition to a quadratic factor may be expressed therefore in the following equivalent forms

$$\left| m_{11} - m_{12} z_i \right| < 1 \quad i = 1, 2 \quad (38)$$

$$\left| 1 - \frac{1}{z_i^{n-2}} \prod_2 (z_i) \right| < 1 \quad i = 1, 2 \quad (39)$$

$$\left| 1 - \left(1 - \frac{z_3}{z_i}\right) \left(1 - \frac{z_4}{z_i}\right) \dots \left(1 - \frac{z_n}{z_i}\right) \right| < 1 \quad (40)$$

$$i = 1, 2$$

The last form in particular will permit an interpretation of the convergence condition as a function of the location of the roots in the complex plane.

Comparison between the Convergence Conditions
Obtained for Linear Factors and Quadratic Factors

Moore and Oldenburger [9] in their study of the convergence of the right to left synthetic division when applied to the search for linear factors found for the convergence condition

$$0 < \frac{z_1 (z_1 - z_2) (z_1 - z_3) \dots (z_1 - z_n)}{z_1^n} < 2 \quad (41)$$

or

$$0 < \left(1 - \frac{z_2}{z_1}\right) \left(1 - \frac{z_3}{z_1}\right) \dots \left(1 - \frac{z_n}{z_1}\right) < 2 \quad (42)$$

This may be rewritten as

$$\left| 1 - \left(1 - \frac{z_2}{z_1}\right) \left(1 - \frac{z_3}{z_1}\right) \dots \left(1 - \frac{z_n}{z_1}\right) \right| < 1 \quad (43)$$

If $P_n(z)$ is expressed as in (39) and $\prod_1(z)$ is defined by

$$\prod_1(z) = \frac{P_n(z)}{(z - z_1)} \quad (44)$$

there results

$$\left| 1 - \frac{1}{z_1^{n-1}} \prod_1(z_1) \right| < 1 \quad (45)$$

Equation (45) is very similar in form to Equation (39) which was determined in the present work for the case of quadratic factors. However, in the case of a linear factor only one condition has to be satisfied at a time (for $z = z_1$) in order to get one root, whereas for a quadratic factor the same condition has to be satisfied simultaneously for two values of z ($z = z_1$ and $z = z_2$).

Assume for instance that the polynomial $P_n(z)$ is written as

$$P_n(z) = (z - z_1)(z - z_2) S_n(z) \quad (46)$$

When the method for linear factors is applied first the condition for convergence is

$$\left| 1 - (z_1 - z_2) S_n(z_1) \right| < 1 \quad (47)$$

so that the first linear factor can be found.

Then in order to get the second the condition becomes

$$\left| 1 - S_n(z_2) \right| < 1 \quad (48)$$

whereas the conditions for extracting the quadratic factor

$$p^*(z) = (z - z_1)(z - z_2) \quad (49)$$

by the quadratic factor method, are

$$\left| 1 - S_n(z_i) \right| < 1 \quad i = 1, 2 \quad (50)$$

It is therefore possible that for some equations both z_1 and z_2 would fail to satisfy Equation (47) whereas they would satisfy Equation (50). Although the method for extracting linear factors would then fail, the method for quadratic factors would allow the user to obtain these factors.

Interpretation of the Convergence Conditions

Oldenburger [8] and Moore and Oldenburger [9] assumed in their studies of the convergence of the right to left synthetic division that all polynomials to which the method was applied represented stable systems in the sense of Routh - Hurwitz and had therefore all their zeros in the left half plane. However, stability was not sufficient for the method to converge.

In the following studies it is going to be shown first that the half plane in which the roots lie does not have anything to do with the convergence of the method and then some particular cases will be considered.

If for instance one considers the convergence condition given by Equation (40)

$$\left| 1 - \left(1 - \frac{z_3}{z_1}\right) \left(1 - \frac{z_4}{z_1}\right) \dots \left(1 - \frac{z_n}{z_1}\right) \right| < 1 \quad (40)$$

$i = 1, 2$

It can be seen that changing the signs of all the real parts of the roots does not change the convergence condition since all complex roots occur in conjugate pairs. In other words the convergence condition is the same for two equations the roots of which are symmetrical with respect to the imaginary axis of the z-plane.

Particular Case of a Cubic

Equation (40) becomes

$$\left| 1 - \left(1 - \frac{z_3}{z_1} \right) \right| < 1 \quad i = 1, 2 \quad (51)$$

or

$$|z_3| < |z_1| \quad i = 1, 2 \quad (52)$$

This can be interpreted as the condition that the two roots forming the quadratic factor which is sought must lie outside the circle with the origin as center and radius $|z_3|$. This indicates that the method will fail for the case in which two of the roots are complex conjugate with moduli smaller than that of the real root. In such cases if the method for real roots does not converge either, one could try the methods exposed further for cases in which the iteration diverges, or use one of the transformations suggested by Moore and Oldenburger [9].

As may be seen above the location of the roots in the z-plane did not intervene as such and only the moduli counted.

It is also to be remarked that the condition of convergence

(52) which was found in that case would also be found if the method for linear factor was applied to find the roots of a quadratic equation.

Particular Case of a Quartic

The condition (40) now yields

$$\left| 1 - \left(1 - \frac{z_3}{z_1} \right) \left(1 - \frac{z_4}{z_1} \right) \right| < 1 \quad (53)$$

which can be written as

$$\left| \frac{z_3 + z_4}{z_1} - \frac{z_3 z_4}{z_1^2} \right| < 1 \quad i = 1, 2 \quad (54)$$

From Equation (54) it can be seen that the domain in which the roots $z_i = z_1$ and $z_i = z_2$ of the quadratic can be chosen depends explicitly on the values of the roots z_3 and z_4 . Since z_3 and z_4 can only be real or complex conjugate, their sum and their product are real.

It can be seen therefore that the domain defined by Equation (54) is symmetrical with respect to the real axis, and a change of z_3 and z_4 to $-z_3$ and $-z_4$ respectively corresponds to a reflection about the imaginary axis.

These domains of convergence have been plotted for each of the cases which might occur in Figures 2 - 4. It can be noticed that in each case the assumption has been made that either the real part of z_3 and z_4 was equal to -1 (in the case when z_3 and z_4 are complex conjugate) or that the smallest of the real roots z_3, z_4 was equal to -1 (in the

case when z_3 and z_4 are real). This assumption is based on the previous remarks about the symmetry involved, and also on the fact that if a polynomial

$$P_n(z) = z^n + a_{n-1} z^{n-1} + a_{n-2} z^{n-2} + \dots + a_1 z + a_0 = 0 \quad (55)$$

has for roots $z_1, z_2, z_3, \dots, z_n$, the polynomial having for roots $\frac{z_1}{k}, \frac{z_2}{k}, \dots, \frac{z_n}{k}$ is obtained by dividing the coefficients a_k of $P_n(z)$ by the increasing powers of k to yield

$$P_n\left(\frac{z}{k}\right) = z^n + \frac{a_{n-1}}{k} z^{n-1} + \frac{a_{n-2}}{k^2} z^{n-2} + \dots + \frac{a_1}{k^{n-1}} z + \frac{a_0}{k^n} = 0 \quad (56)$$

An example will be shown afterwards to illustrate how the curves can be used.

The possible cases corresponding to the different locations of the roots z_3, z_4 in the complex half planes are considered next.

Complex Conjugate Roots

Let $z_3, z_4 = -1 \pm jy$. The curves determining the zone of convergence have been plotted on Figure 2 for $y = 0, 0.1, 0.5, 1$ and 2 .

The method can converge to the quadratic formed by the other set of roots (z_1, z_2) if those are outside the domain

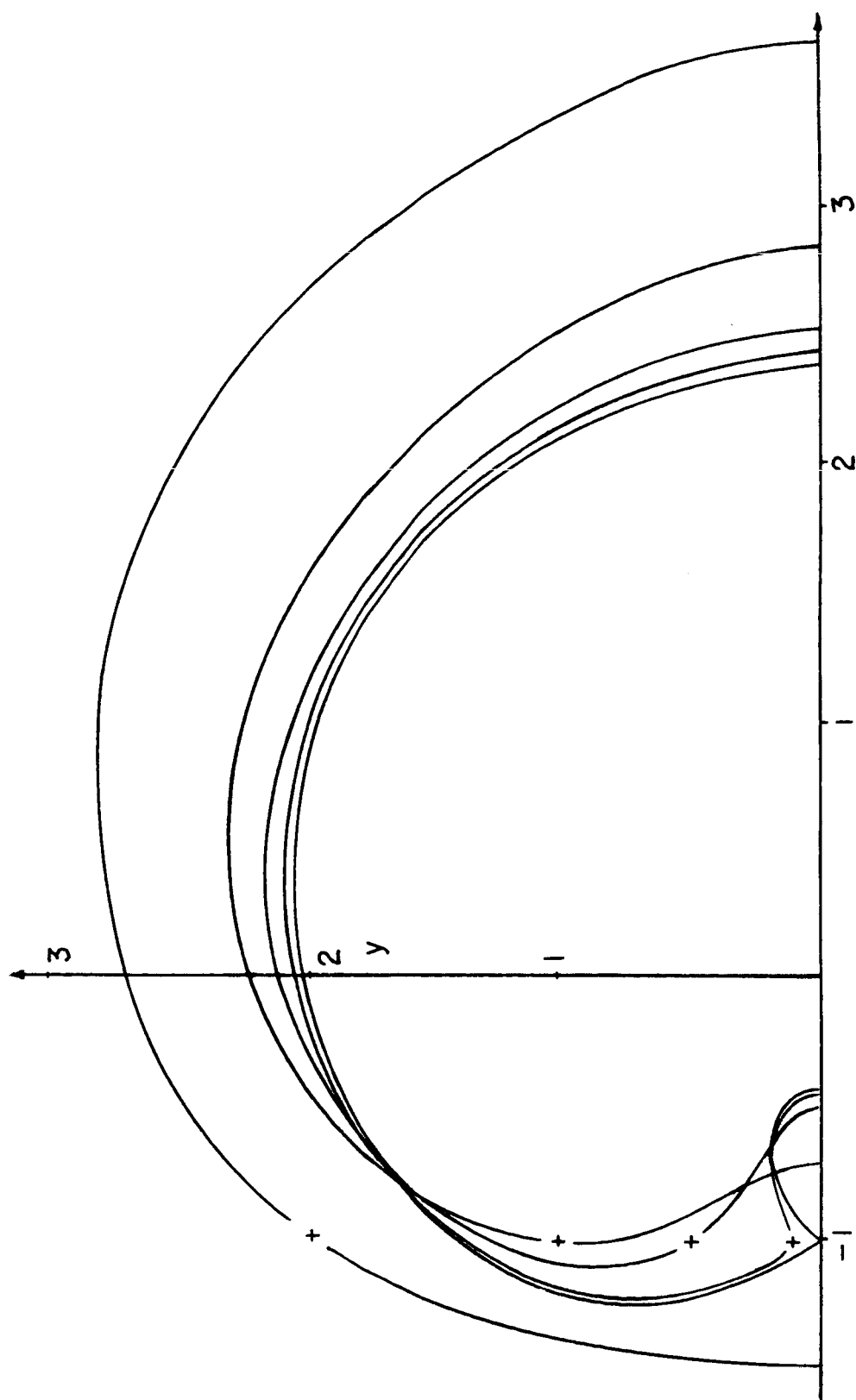


Figure 2
Convergence Regions for Pairs of Complex Conjugate Roots
in the Case of a Quartic

enclosed by the curve.

Real Roots of the Same Sign

Let $z_3 = -1$; $z_4 = -x$; $x > 1$. The corresponding curves have been plotted on Figure 3 for $x = 1$ and $x = 2$.

The method can converge if the roots of the other quadratic are outside the domain situated between the two branches of each curve. However, in practical applications the inner loop, the size of which decreases as x increases, will not yield good convergence and is more unlikely to be met. This loop will always lie between -1 and 0 and always pass through the -1 point.

Roots Real and of Opposite Signs

Let $z_3 = -1$, $z_4 = x$, $x > 1$. The curves have been plotted in Figure 4 for $x = 2, 5, 3 + 2\sqrt{2}, 6, 8$.

The same remarks as in the previous case are still valid except that in this case the curve will consist of one branch for $x \leq 3 + 2\sqrt{2}$ and of two branches for $x > 3 + 2\sqrt{2}$. When two branches occur, the inner loop will always lie on the outside of the segment defined by the point -1 and the origin, and pass through the point -1 .

Although the second and third cases show that the method can be used to extract quadratic factors composed of two real roots, in practice, it is often more practical to try to extract these real roots earlier in the computation if this does not require too many calculations.

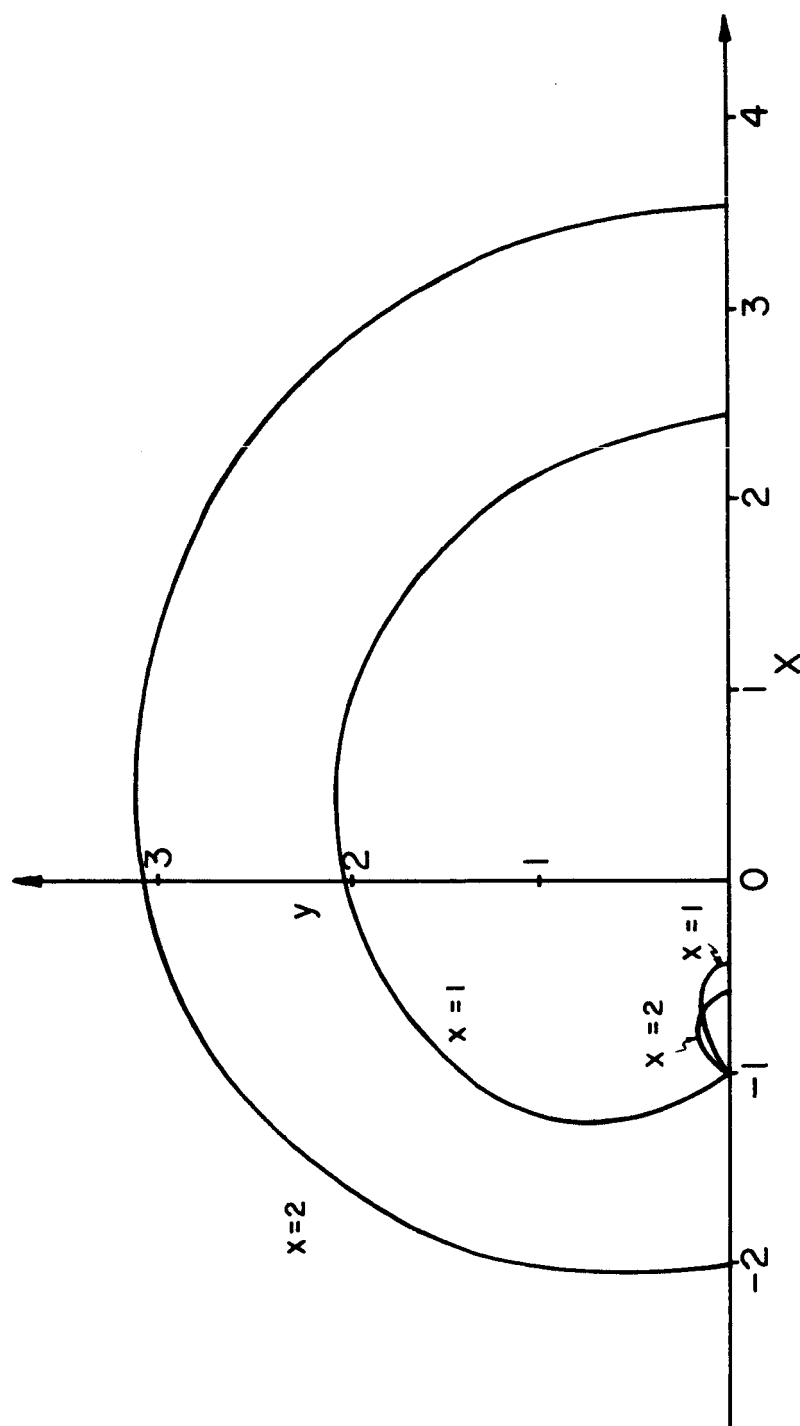


Figure 3

Convergence Regions for Pairs of Real Roots of
Same Sign in the Case of a Quartic

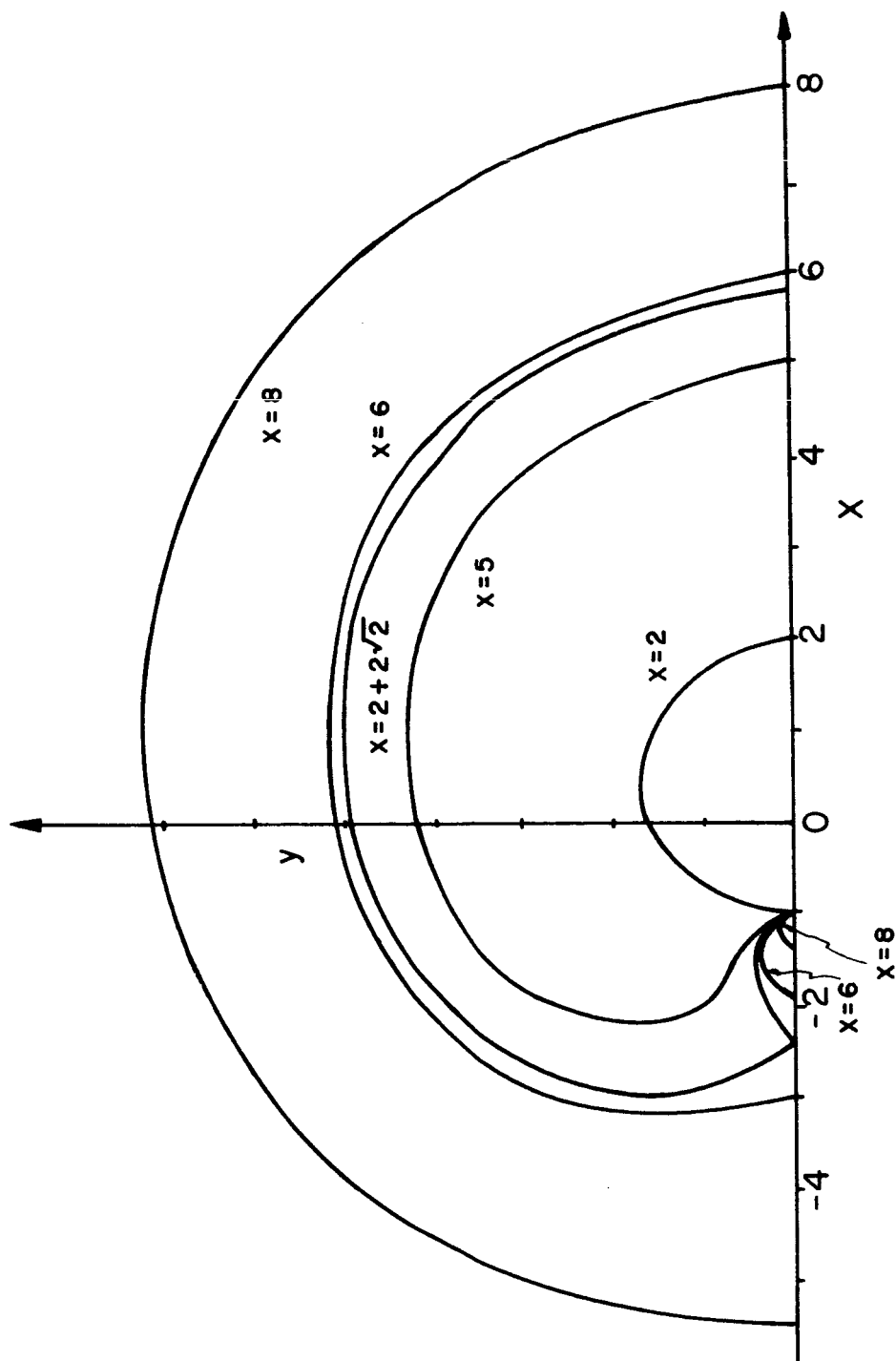


Figure 4

Convergence Region for Pairs of Real Roots of
Opposite Signs in the Case of a Quartic

It can also be noticed that the curves which have been determined always pass through the z_3 and z_4 points, whether real or complex.

The intersections with the real axis are given by the real roots of

$$z^2 - (z_3 + z_4) z + z_3 z_4 = 0 \quad (57)$$

and of

$$z^2 + (z_3 + z_4) z - z_3 z_4 = 0 \quad (58)$$

It can be noticed that at least one of the Equations (57) and (58) has real roots, and that the roots of Equation (57) are z_3 and z_4 .

Examples

If a quartic has the complex conjugate roots $-2 \pm j1$ and $-2 \pm j2$, it can be seen from Figure 2, multiplying the scales by a factor 2 that the method can converge on the pair $-2 \pm j1$ but cannot converge on $-2 \pm j2$, because the first pair is outside the domain of divergence of the second and the second inside that determined by the first.

On the other hand, the method would fail for $-1 \pm j0.5$ and $1 \pm j1$ because each pair would be in the zone of divergence defined by the other one. (In that case, as seen earlier, the zone of divergence corresponding to $1 \pm j1$ is obtained by taking in Figure 2 the symmetric with respect to the imaginary axis of the zone defined by $-1 \pm j1$).

Considering again the curves of Figure 2, it can be seen that if all roots are complex and on the same side of the axis it is unlikely, although possible, that both pairs of roots will lie inside the domains of divergence defined by the other pair.

This explains the recommendation made in earlier papers by Oldenburger [8], and Moore and Oldenburger [9] that all roots should lie in the same half-plane. Therefore it might be worthwhile in the case of divergence when the method is applied to a quartic or to a higher order equation to make a translation which would bring them in the same half-plane. The details of the translation have been explained in a previous work by Moore and Oldenburger [9]. (It makes no difference which half-plane is chosen.)

It is, however, possible to find examples such as the one illustrated in Figure 5 in which both sets of complex roots ($-1 \pm j0.1$ and $-0.75 \pm j1.5$) are in the same half-plane and inside the region of divergence defined by the other pair.

It is also to be noted that the conditions for convergence to a quadratic factor which were found in Equations (53) and (54) are identical to those which would be obtained if convergence to a linear factor were sought when the initial polynomial is a cubic.

The same study could be made for equations of higher order, but the results become more complicated and do not yield any simple geometric interpretation.

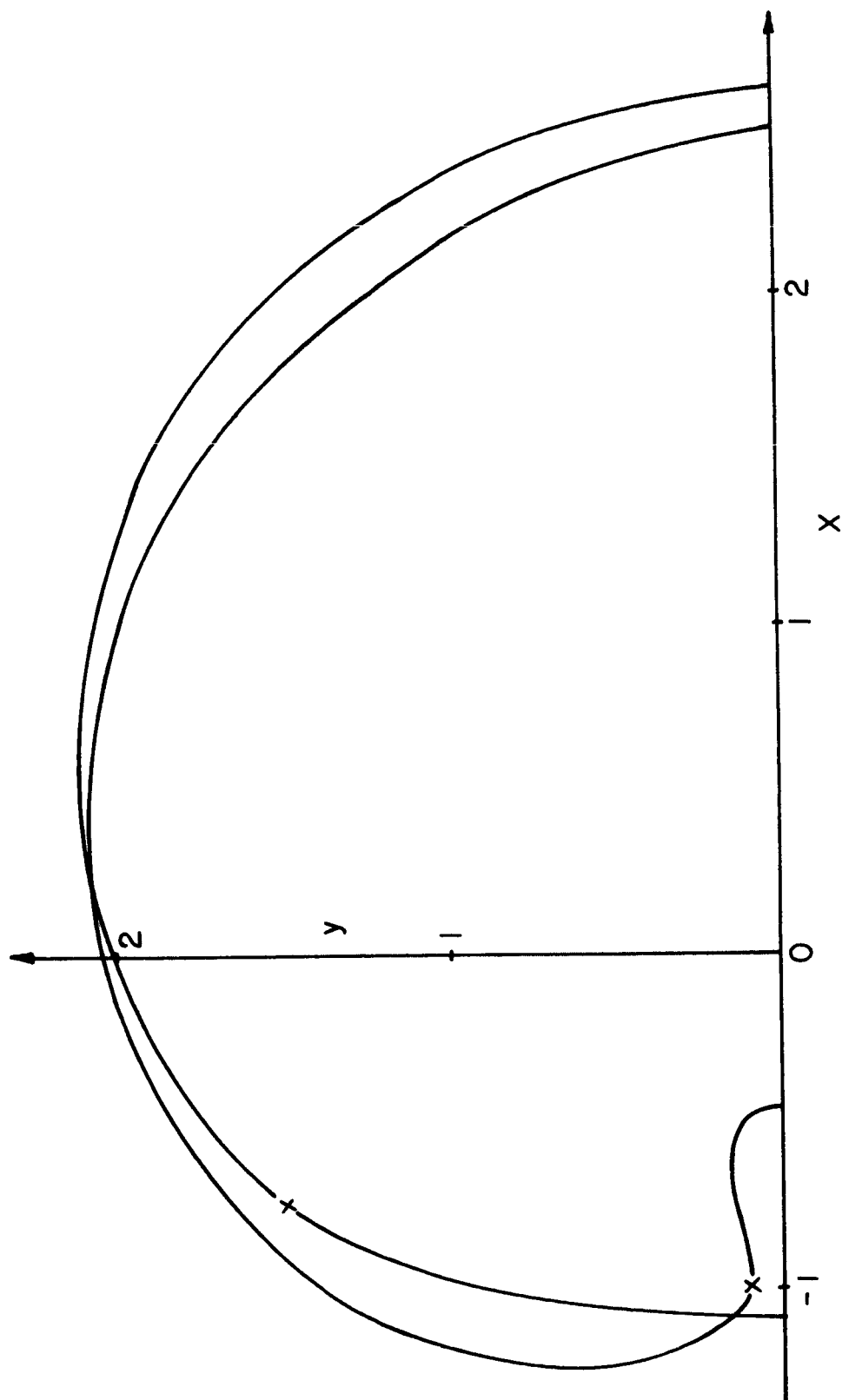


Figure 5

Convergence Regions for the Case of the Quartic
of Roots $-1 \pm j0.1$ and $-0.5 \pm j1.5$

2. ANALYSIS OF THE CONVERGENCE OF THE ITERATION PROCESS

It has been shown in the previous chapter that the values of x_1 and x_2 at two successive steps of the iteration were related by the expression

$$\begin{bmatrix} x_1, u+1 \\ x_2, u+1 \end{bmatrix} = \begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix} \begin{bmatrix} x_1, u \\ x_2, u \end{bmatrix} \quad (59)$$

or

$$X_{u+1} = M X_u \quad (60)$$

Equation (60) can be interpreted as a relation between the successive "error vectors" in the X-plane which is the same for a translation of the axes as the α, β -plane.

In order to gain insight concerning the convergence of the iteration, it is advantageous to consider first the normal form of Equation (59). Defining

$$X = P Y \quad (61)$$

it is possible to determine the matrix

$$P = \begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix} \quad (62)$$

such that

$$Y_{u+1} = \mathcal{A} Y_u \quad (63)$$

where

$$\mathcal{A} = P^{-1} M P \quad (64)$$

and

$$\mathcal{A} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \quad (65)$$

λ_1 and λ_2 being the characteristic roots of matrix M and P^{-1} the inverse of P , i.e.

$$P P^{-1} = I \quad (66)$$

where I is the unity matrix.

The transformation from the X coordinates to the Y coordinates represent a skewing of the reference axes. The analysis of the convergence is greatly facilitated in the new system.

Equation (63) defines the iteration in the Y coordinates. If applied k times in succession it yields

$$Y_{k+u} = \mathcal{A}^k Y_u \quad (67)$$

or

$$\begin{bmatrix} y_{1, k+u} \\ y_{2, k+u} \end{bmatrix} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}^k \begin{bmatrix} y_{1, u} \\ y_{2, u} \end{bmatrix} \quad (68)$$

In order to interpret Equation (68) different cases have to be considered depending upon the nature and the relative magnitudes of the λ 's.

It was shown earlier that the characteristic roots of M were given by

$$\lambda_i = m_{11} - m_{12} z_i; i = 1, 2 \quad (69)$$

where z_i represents either root of $p^*(z)$ and m_{11} and m_{12} are given by Equations (26) and (27).

Since B , m_{11} and m_{12} are real, a sufficient condition for the λ 's to be real is that the roots of $p^*(z)$ be real. This condition is not necessary, however, because real λ 's can be obtained in the case of complex conjugate z_i 's whenever $m_{12} = 0$.

The condition $m_{12} = 0$ yields two equal λ_i 's whether the z_i 's are real or complex. Equal λ_i 's are also obtained in the obvious case $z_1 = z_2$.

Finally the last condition yielding a particular case occurs whenever

$$\frac{A}{B} = \frac{2m_{11}}{m_{21}} \quad (70)$$

which corresponds to the case $\lambda_1 = -\lambda_2$.

In order to make a complete study of the convergence of the iteration process, each case will be considered separately.

Real Characteristic Roots

When the characteristic roots of the matrix M are real, Equation (68) can be rewritten

$$y_{1, k+u} = \lambda_1^k y_{1,u} \quad (71)$$

$$y_{2, k+u} = \lambda_2^k y_{2,u}$$

Equation (71) shows that the components y_1 and y_2 of Y converge or diverge independently of each other and illustrate the necessity of the convergence condition given earlier, that is

$$\left| \lambda_{\max} \right| < 1 \quad (72)$$

Since y_1 and y_2 define a new system of reference axes, it might be interesting to locate them with respect to the X -axes and the α, β -axes.

Equation (64) can be rewritten

$$P \mathcal{A} = M P \quad (73)$$

Replacing P , $\mathcal{A}$ and M by their values given by Equations (62), (65) and (59) the following four simultaneous equations are obtained

$$\begin{aligned}
p_{11} \lambda_1 &= m_{11} p_{11} + m_{12} p_{21} \\
p_{21} \lambda_1 &= m_{21} p_{11} + m_{22} p_{21} \\
p_{12} \lambda_2 &= m_{11} p_{12} + m_{12} p_{22} \\
p_{22} \lambda_2 &= m_{21} p_{12} + m_{22} p_{22}
\end{aligned} \tag{74}$$

Whereupon, considering successively the first two and the last two Equations (74) there follows

$$\begin{aligned}
\frac{p_{21}}{p_{11}} &= \frac{\lambda_1 - m_{11}}{m_{12}} = \frac{m_{21}}{\lambda_1 - m_{22}} = s_1 \\
\frac{p_{22}}{p_{12}} &= \frac{\lambda_2 - m_{11}}{m_{12}} = \frac{m_{21}}{\lambda_1 - m_{22}} = s_2
\end{aligned} \tag{75}$$

From Equations (75) the elements of the P-matrix can be determined within constant factors, giving for Equation (62)

$$P = \begin{bmatrix} p_{11} & p_{12} \\ s_1 p_{11} & s_2 p_{12} \end{bmatrix} \tag{76}$$

and the axes y_1 ($y_2 = 0$) and y_2 ($y_1 = 0$) of the Y-plane become in the X-plane the lines

$$\begin{aligned}
x_2 &= s_1 x_1 & (y_2 = 0) \\
x_2 &= s_2 x_1 & (y_1 = 0)
\end{aligned} \tag{77}$$

where s_1 and s_2 are determined by Equations (75).

The values s of the slopes of the Y-axes in the α, β -plane can be related easily to the roots of the quadratic factor $p^*(z)$. Substituting in Equations (75) the values of λ_i given by Equation (69) there results

$$s_1 = -\frac{m_{12} z_1}{m_{12}} \quad (78)$$

$$s_2 = -\frac{m_{12} z_2}{m_{12}}$$

if $m_{12} \neq 0$, there follows

$$s_1 = -z_1 \quad (79)$$

$$s_2 = -z_2$$

This remarkably simple result means that whenever Oldenburger's right to left synthetic division is applied in order to determine the coefficients A, B of a quadratic factor

$$p^*(z) = z^2 + Az + B$$

the axes corresponding to the normal equation in the α, β -plane are defined as the lines passing through A, B and having for slopes the negative of the roots of $p^*(z)$. The only restrictions are that $m_{12} \neq 0$ and the λ 's are real. The cases which do not satisfy these conditions are considered

separately later on.

Real Characteristic Roots of Different Magnitudes

The characteristic roots of M are real and have different magnitudes whenever

$$\begin{aligned} z_i \quad (i = 1, 2) \text{ real} \\ z_1 \neq z_2 \\ m_{12} \neq 0 \\ \frac{A}{B} \neq \frac{2m_{11}}{m_{21}} \end{aligned} \tag{80}$$

The locus of the points of the iteration takes on different forms depending upon the signs of λ_1 and λ_2 and their relative magnitudes. It may be seen from Equation (71) that the component corresponding to the smaller λ in absolute value decreases faster. Therefore the iteration converges to the final value A, B and the curve locus of the points is tangent to the axis corresponding to the larger λ . The rate of convergence is determined by the larger λ in absolute value.

Positive λ_1 and λ_2

The case of λ_1 and λ_2 both positive, is of particular interest because it is the only one which does not yield "oscillatory convergence", i.e. the successive points of the

iteration converge monotonically towards the point A,B according to Equations (71) remaining in the same quadrant of the plane of the Y-axes once they are close enough to the final value so that the linear approximation is valid. The curve on which the successive points of the iteration lie reaches A,B, tangent to the axis corresponding to $\lambda_{\max}$.

The convergence in the general case of λ_1 and λ_2 both positive is illustrated in Figure 6i.

Negative λ_1 and λ_2

In the case of λ_1 and λ_2 both negative, the components y_1 and y_2 of Y_k change their signs at every iteration. The points Y_k lie therefore alternatively on either side of the final value on a curve the two branches of which become tangent at A,B to the axis associated with the λ having the larger absolute value. It is to be noted that the curve crosses this axis at A,B and turns its concavity in opposite directions on both sides of that point. The rate of convergence is here also determined by the λ of larger absolute value.

The case of two λ 's of negative signs is illustrated in Figure 6ii.

Real Characteristic Roots of Opposite Signs

Since one of the components of Y_k changes sign at each iteration, the locus of the points of the iteration has two

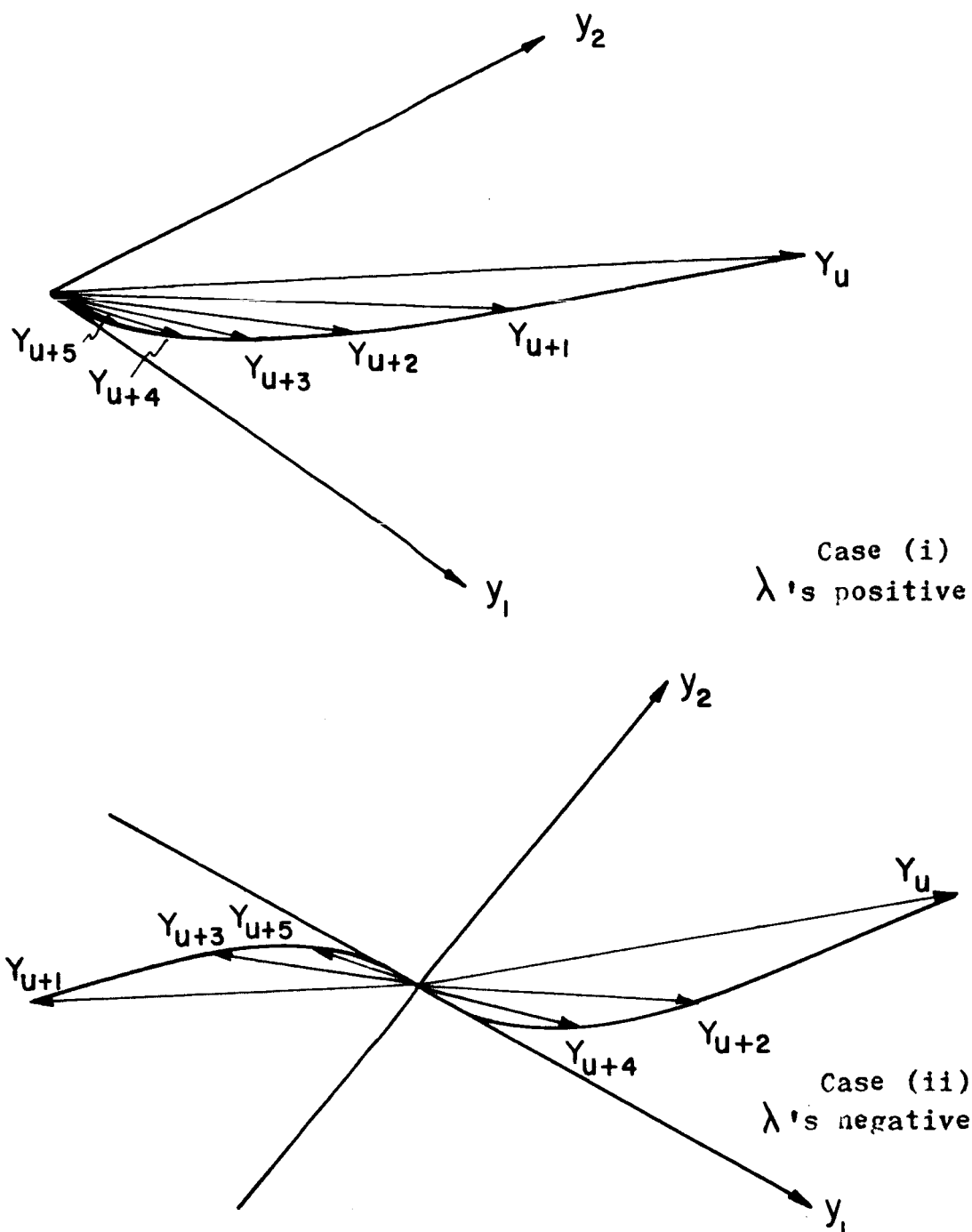


Figure 6
 Successive Error Vectors in the Case of Real
 Characteristic Roots of Same Sign

branches coming to the point A,B. These branches are tangent to the axis corresponding to the larger λ in absolute value.

Positive λ Greater in Absolute Value than Negative λ

The two branches of the curve on which the Y_k points lie, approach A,B tangent to and on opposite sides of the semi-axis corresponding to λ_{pos} . This is illustrated in Figure 7i.

Negative λ Greater in Absolute Value than Positive λ

The two branches of the curve on which the Y_k points lie approach A,B tangent to the axis corresponding to λ_{neg} . All the points of the iteration lie on the same side of this axis but alternatively on either side of the semi-axis corresponding to λ_{pos} . This is illustrated in Figure 7ii.

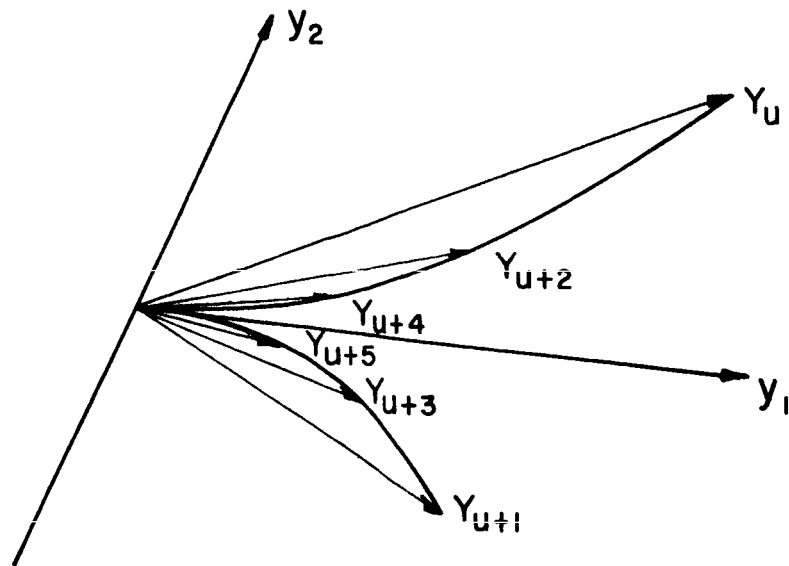
Real Characteristic Roots Equal in Magnitude

It has been seen previously that real λ 's of equal magnitude can occur under different circumstances. Each of them has to be considered separately for it yields a different type of convergence.

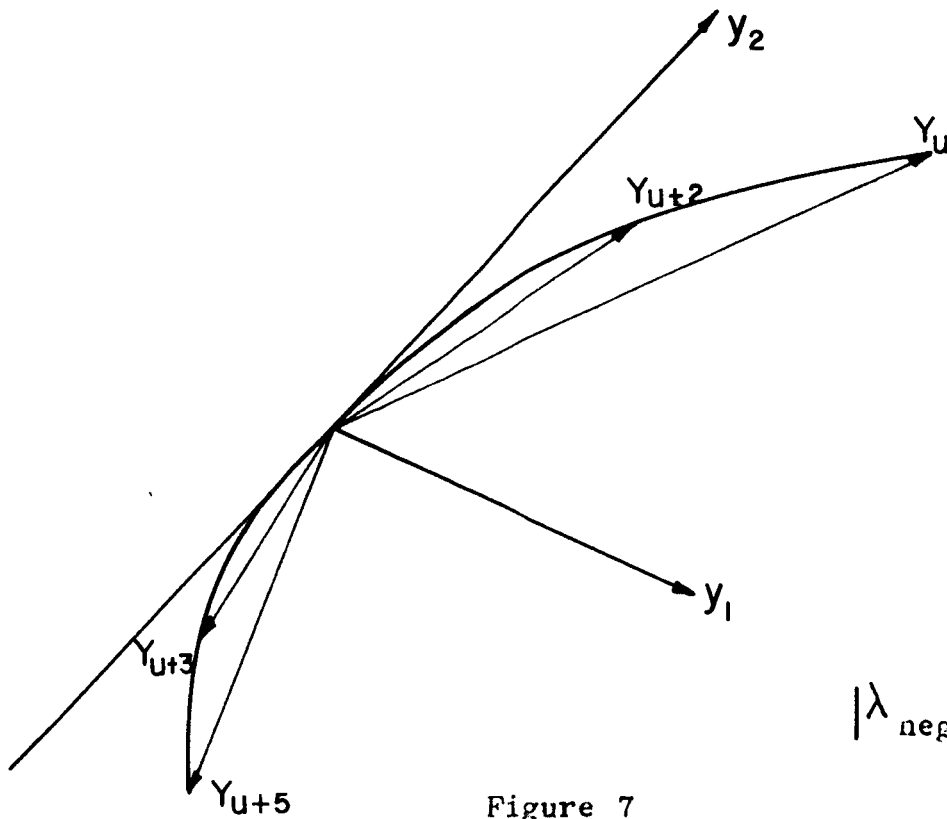
Equal Real Roots and $m_{21} \neq 0$

In order to obtain real equal roots when $m_{12} \neq 0$ it is necessary and sufficient for the roots of $p^*(z)$ to satisfy the relation

$$z_1 = z_2$$



Case (i)
 $|\lambda_{\text{pos}}| > |\lambda_{\text{neg}}|$



Case (ii)
 $|\lambda_{\text{neg}}| > |\lambda_{\text{pos}}|$

Figure 7

Successive Error Vectors in the Case of Real
 Characteristic Roots of Opposite Signs

It can be seen from Equations (78) and (80) which still apply when $z_1 = z_2$ and $m_{12} \neq 0$, that two Y-axes have the same slope and therefore form a same line which has for slope the negative of the common root.

The convergence happens in a way similar to that of the first two considered cases depending upon the sign of the λ 's. The fact that the two Y-axes form one line does not affect the convergence; it is only necessary to remember that these axes correspond to the linear approximation and the first order terms. In practice the second order terms are also taken into account when the iteration is made, and, as long as they are negligible, they will bring the successive points of the iteration very quickly to the vicinity of the common axes.

The two possible cases which arise whenever $z_1 = z_2$ and $m_{21} \neq 0$ are considered next.

Both λ 's Equal and Positive, $m_{21} \neq 0$

The case of both λ 's equal and positive with $m_{21} \neq 0$ is simply an extension of the first case considered and illustrated in Figure 61. It yields a non-oscillatory convergence and the curve locus of the points of the iteration is tangent at A,B to the common axis.

Both λ 's Equal and Negative, $m_{21} \neq 0$

The case of two equal negative λ 's resulting from equal roots of the quadratic factor $p^*(z)$, is comparable to the

general case illustrated in Figure 6 ii in which the two λ 's are different and negative. In this case although the two axes y_1 and y_2 are not distinct either.

Equal Real Roots in the Case $m_{21} = 0$

It has been shown earlier that equal λ 's are obtained whenever the relation

$$m_{21} = 0 \quad (81)$$

is satisfied, whether the roots of the quadratic factor $p^*(z)$ are real or conjugate complex.

Equations (78) are then not valid to obtain the slopes of the Y-axes since they yield an indeterminate form zero over zero.

However, since Equation (81) entails also

$$m_{12} = 0 \quad (82)$$

Equation (59) becomes:

$$\begin{bmatrix} x_{1, u+1} \\ x_{2, u+1} \end{bmatrix} = \begin{bmatrix} m_{11} & 0 \\ 0 & m_{22} \end{bmatrix} \begin{bmatrix} x_{1, u} \\ x_{2, u} \end{bmatrix} \quad (83)$$

or

$$\begin{bmatrix} x_{1, u+1} \\ x_{2, u+1} \end{bmatrix} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \begin{bmatrix} x_{1, u} \\ x_{2, u} \end{bmatrix} \quad (84)$$

Equation (59) is therefore in the normal form and the Y-axes are the same as the X-axes.

Unlike the case $m_{21} \neq 0$, the convergence in the case of equal λ 's and $m_{21} = 0$ introduces new aspects.

Equation (84) can be rewritten since the λ 's are equal

$$\begin{bmatrix} x_{1, u+1} \\ x_{2, u+1} \end{bmatrix} = \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_{1, u} \\ x_{2, u} \end{bmatrix} \quad (85)$$

which yields when the iteration is applied k times in a row

$$\begin{bmatrix} x_{1, u+k} \\ x_{2, u+k} \end{bmatrix} = \lambda^k \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_{1, u} \\ x_{2, u} \end{bmatrix} \quad (86)$$

which is equivalent to

$$\begin{aligned} x_{1, u+k} &= \lambda^k x_{1, u} \\ x_{2, u+k} &= \lambda^k x_{2, u} \end{aligned} \quad (87)$$

The coordinates of the successive points remain always in the same ratio; the points can therefore be expected to lie on one or two straight lines depending upon the sign of λ , passing through the point A,B and determined by the first trial factor.

The two possible cases are considered next.

Both λ 's are Positive, $m_{21} = 0$

Similarly to the other cases for which both the λ 's were positive the convergence occurs in a non-oscillatory way.

The main difference between this case and the previous ones is the absence of an asymptotic direction; the slope of the tangent to the curve locus of the points of the iteration at A,B depends upon the initial trial value.

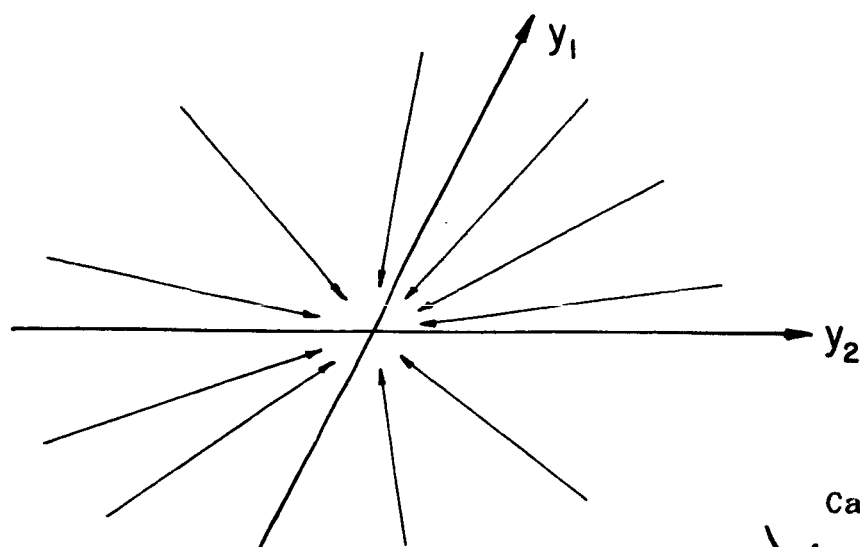
In the theoretical case for which the linear approximation would be valid all over the α, β -plane, all the points of the iteration would lie on the straight line passing through the final and the initial points. This is illustrated in Figure 8i.

In practice, however, the linear approximation is only valid in a certain neighborhood of the final value. The tangent at A,B to the curves loci of the points of the iteration still depends upon the initial value insofar that these curves reach A,B from different directions for different initial values and these curves are generally linear in a close neighborhood of A,B.

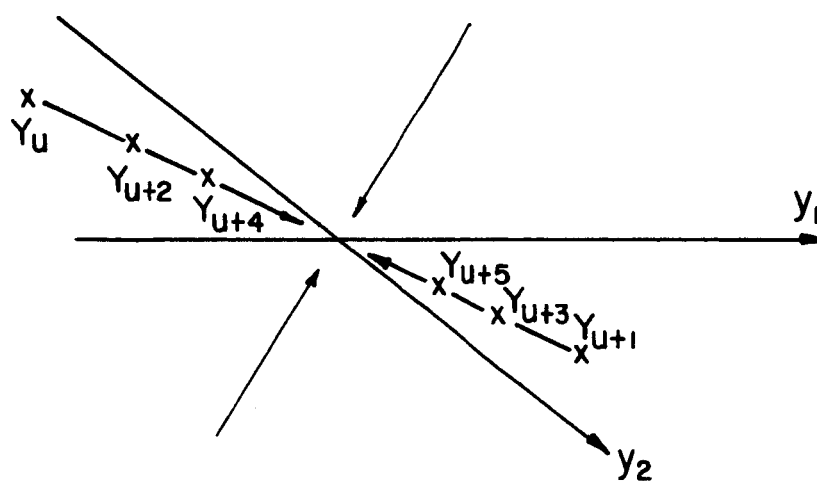
Both λ 's are Negative, $m_{21} = 0$

In the case of both λ 's negative and $m_{21} = 0$ an oscillatory convergence is obtained, however in this case as in the previous one there is no common tangent at A,B to the curves loci of the iteration.

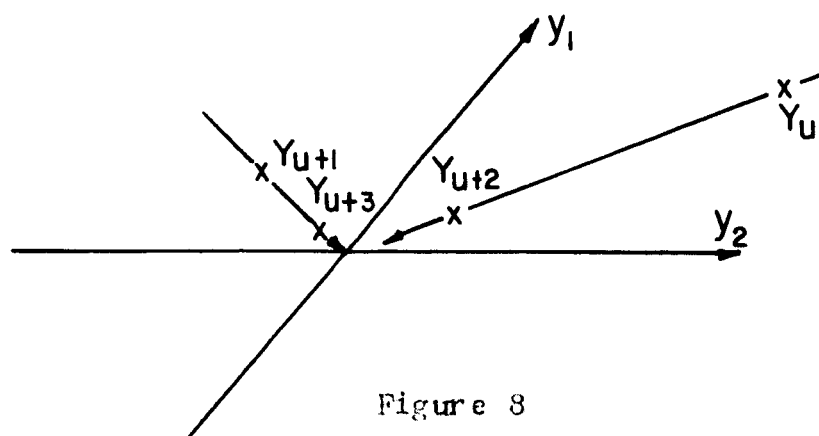
In the idealized case shown in Figure 8ii, for which



Case (i)
 λ 's positive



Case (ii)
 λ 's negative



Case (iii)
 $\lambda_1 = -\lambda_2$

Figure 8

Convergence Patterns for the Case of Real
Characteristic Roots Equal in Magnitude

the linear approximation is assumed to be valid throughout the plane, it can be seen that the points of the iteration lie alternatively on either side of the final value on the straight line determined by the first trial factor and the final value.

In practice the terms of second order in $\frac{x_1}{A}$ and $\frac{x_2}{B}$ have also some influence on the behaviour of the iteration and the curve will not be a straight line anymore except in the neighborhood of the exact value.

Characteristic Roots Equal in Magnitude
but Opposite in Signs

The case of $\lambda_1 = -\lambda_2$ is obtained whenever

$$\frac{A}{B} = \frac{2m_{11}}{m_{21}} \quad (88)$$

This case is a limit case between the two cases considered previously and for which either the positive or the negative λ had the larger absolute value. Equation (59), once put in the normal form may be written

$$\begin{bmatrix} y_{1, u+1} \\ y_{2, u+1} \end{bmatrix} = \lambda \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} y_{1, u} \\ y_{2, u} \end{bmatrix} \quad (89)$$

Reasoning as was done in the case $m_{21} = 0$, it is found that in the linearized case, the locus of Y_k consists of two

straight lines determined by the initial trial factor, the result of the first iteration and the final value. The two lines lie on the same side of the axis which corresponds to the negative λ , but on opposite sides of the axis corresponding to the positive λ . This is illustrated in Figure 8iii.

In practice, as in the cases $m_{21} = 0$, the linear domain is limited and only in a certain neighborhood of the point A,B the curves can be compared to straight lines.

Complex Conjugate Characteristic Roots

It has been shown earlier that the necessary and sufficient condition for the characteristic roots of the matrix M to be complex conjugate is

$$\begin{aligned} z_1 \text{ and } z_2 \text{ complex conjugate} \\ m_{21} \neq 0 \end{aligned} \quad (90)$$

Denoting λ_1 and λ_2 by

$$\begin{aligned} \lambda_1 &= \rho e^{j\theta} ; \lambda_1 = \mu + j\nu \\ \lambda_2 &= \rho e^{-j\theta} ; \lambda_2 = \mu - j\nu \end{aligned} \quad (91)$$

Equation (68) may be rewritten

$$\begin{bmatrix} y_{1, k+u} \\ y_{2, k+u} \end{bmatrix} = \rho^k \begin{bmatrix} e^{jk\theta} & 0 \\ 0 & e^{-jk\theta} \end{bmatrix} \begin{bmatrix} y_{1, u} \\ y_{2, u} \end{bmatrix} \quad (92)$$

From Equation (92) it may be seen that the ratio of the two components of the vectors Y_k remains constant. However at each iteration these components rotate with equal angles but in opposite directions. The vector Y_k which is the resultant also rotates around the final value. The case of complex conjugate λ 's has been illustrated in Figure 9. It can be seen that if K is the smallest integer such that

$$\frac{2\pi}{\theta} \leq K \quad (93)$$

K iterations at least are necessary to describe a portion of the curve corresponding to a rotation of 2π , where θ is determined by

$$\sin \theta = \frac{\nu}{\sqrt{\nu^2 + \mu^2}} \quad ; \quad \cos \theta = \frac{\mu}{\sqrt{\nu^2 + \mu^2}} \quad (94)$$

Furthermore, the ratio of the distances to the point A,B of two points of the curve separated by an angle 2π is equal to $\frac{2\pi}{\theta}$.

Therefore two factors affect the convergence; the smaller ρ , the faster the convergence since $\rho = |\lambda|$, the larger θ , the more spirals the curve locus of the points of the iteration describes before reaching the point A,B. A large θ proves to be very useful for accelerating the search for the roots of slowly converging iterations as will be shown in the next chapter.

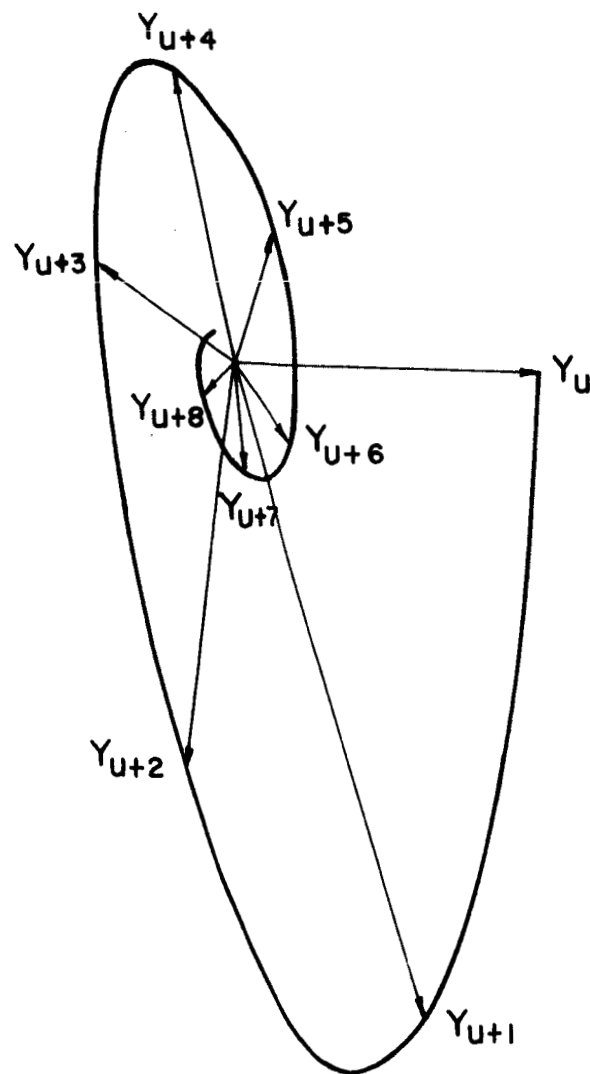


Figure 9
 Successive Error Vectors in the Case of
 Two Complex Conjugate Roots

Examples

In order to illustrate the previous results a few examples are going to be considered next.

Example 1: Two positive λ 's, $\lambda_1 \neq \lambda_2$.

Consider the quartic

$$z^4 + 11z^3 + 43z^2 + 75z + 50 = 0 \quad (95)$$

$$(z^2 + 4z + 5)(z^2 + 7z + 10) = 0 \quad (96)$$

$$(z + 2 + j)(z + 2 - j)(z + 2)(z + 5) = 0 \quad (97)$$

The values of the λ 's may be computed from Equation (36). They are given for the real roots by

$$\lambda = -\frac{1}{z_i^2} (4z_i + 5) \quad (98)$$

which yields

$$\lambda_{-5} = 0.75 ; \lambda_{-2} = 0.6 \quad (99)$$

Similarly for the imaginary roots

$$\lambda = -\frac{1}{z_i^2} (7z + 10) \quad (100)$$

that is:

$$\lambda_{-2 \pm j} = \frac{8 \mp j}{5} \quad (101)$$

As is seen from Equation (101) $\lambda_{-2 \pm j}$ does not satisfy the condition (32) but λ_{-2} and λ_{-5} are positive and smaller

Table 1
Results of the Iteration for the Quartic

$$z^4 + 11z^3 + 43z^2 + 75z + 50 = 0$$

Iteration	α	β
0	11.000	43.00
1	9.553	25.92
2	8.818	20.22
3	8.369	17.33
4	8.066	15.56
5	7.845	14.34
6	7.678	13.47
7	7.546	12.76
8	7.440	12.21
9	7.353	11.78
10	7.282	11.41
11	7.224	11.12
12	7.177	10.89
13	7.139	10.69
14	7.108	10.53
15	7.083	10.41
16	7.064	10.32
17	7.049	10.24
18	7.037	10.19
19	7.028	10.14
20	7.021	10.11
21	7.016	10.08
22	7.012	10.06
23	7.009	10.05
24	7.007	10.03
25	7.005	10.03
26	7.004	10.02
27	7.003	10.01
28	7.002	10.01
29	7.002	10.01
30	7.001	10.01
31	7.001	10.00
32	7.001	10.00
33	7.001	10.00
34	7.000	10.00

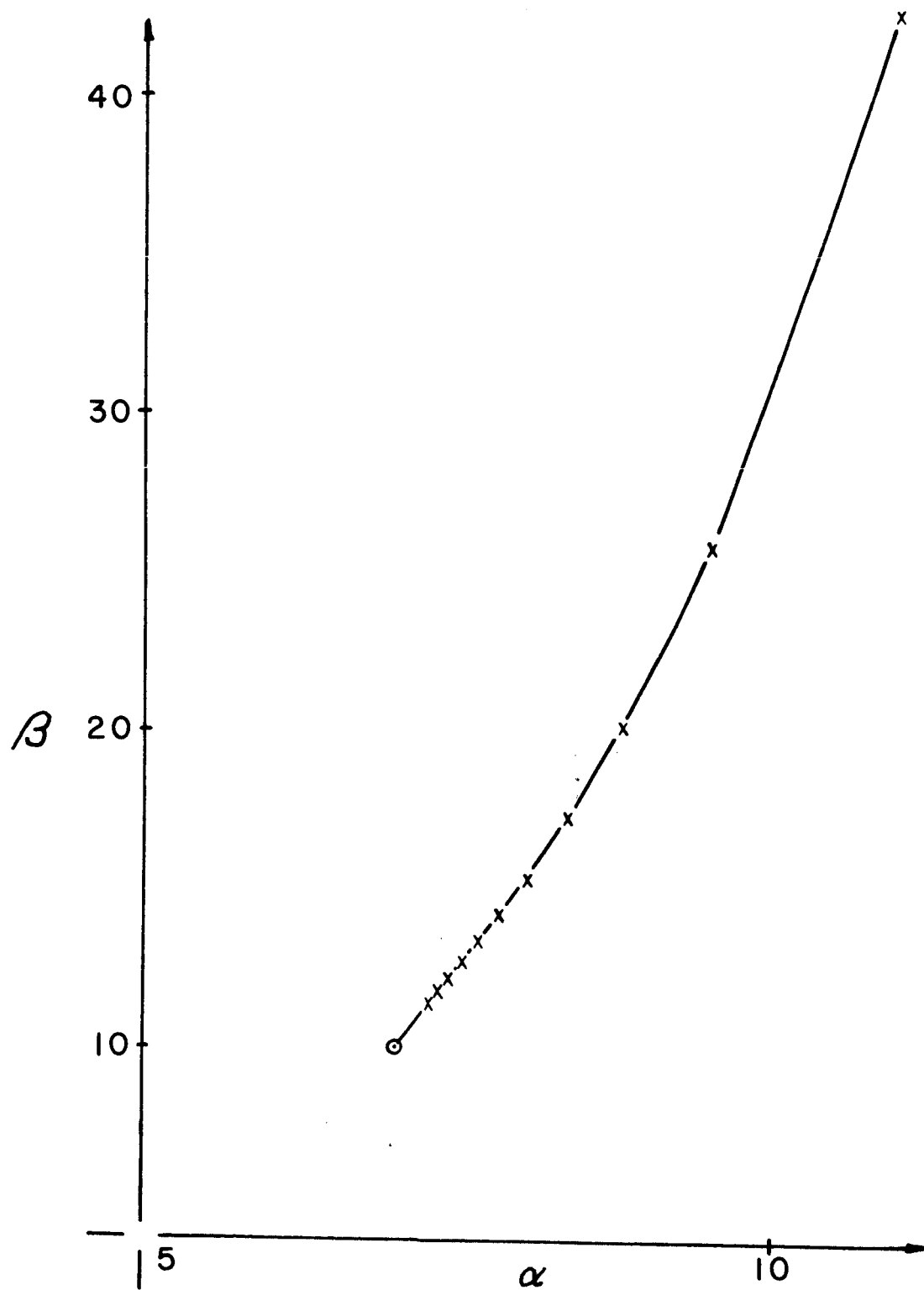


Figure 10

Successive Points of the Iteration for the Quartic

$$z^4 + 11z^3 + 43z^2 + 75z + 50 = 0$$

than one in absolute value. Therefore the iteration converges toward the 7, 10 point of the α, β -plane in a non-oscillatory way and with a $\lambda_{\max} = 0.75$. The results of the iteration are shown in Table 1 and Figure 10.

The Y-axes are the lines passing through the point 7, 10 and the slopes of which are according to Equation (79)

$$s_{-2} = 2 ; s_{-5} = 5 \quad (102)$$

Since λ_{-5} has the larger absolute value, the curve locus of the iteration converges to 7, 10 tangent to the axis of slope $s_{-5} = 5$ as can be checked from Figure 10.

In this example and in the following ones the first trial quadratic was obtained by taking the coefficients of the terms of power $n-1$ and $n-2$ of the given polynomial. This was done in order to have an "unbiased" first guess although the user is advised to choose values of α_0 and β_0 as close as possible to the searched values. When nothing is known about the order of magnitude of these values, the rule of the thumb given by Oldenburger that the first choice be approximately $\frac{1}{2}$ the values of the $n-1$ and $n-2$ terms of $P_n(x)$, yields usually good results.

Example 2: Two negative λ 's, $\lambda_1 = \lambda_2$ ($m_{21} \neq 0$)
Consider the quartic

$$z^4 - 10z^3 + 20z^2 - 24z + 238 = 0 \quad (103)$$

$$(z^2 + 12z + 36)(z^2 + 2z + 8) = 0 \quad (104)$$

$$(z - 6)(z - 6)(z + 1 + j\sqrt{7})(z + 1 - j\sqrt{7}) = 0 \quad (105)$$

The λ 's are found to be

$$\lambda_6 = -\frac{5}{9} \quad (106)$$

$$\lambda_{-1 \pm j\sqrt{7}} = \frac{274 \mp 102j\sqrt{7}}{64} \quad (107)$$

As may be seen $|\lambda_{-1 \pm j\sqrt{7}}| > 1$ but λ_6 is real negative and smaller than one in absolute value. Therefore the iteration can converge toward the point -12, 36 of the α, β -plane with the successive points lying alternatively on either side of the actual value, as may be seen from Table 2 in which the results have been tabulated for the iteration starting at -10, 20. The curve locus of the points of the iterations have been plotted in Figure 11 for a few initial values, to show that all iterations reach the final value with the same slope, which is that of the common axis, i.e. $s = -6$.

It can also be seen from Table 2 that the errors on two successive terms of the iteration are opposed in sign and in a ratio

$$|\lambda| = \frac{5}{9} \quad (108)$$

For two successive points on the same side of the final value the ratio is approximately

$$(\lambda)^2 = \frac{25}{81} \quad (109)$$

Example 3: Two λ 's opposite in sign but equal in magnitude

Table 2
Results of the Iteration for the Quartic

$$z^4 - 10z^3 + 20z^2 - 24z + 288 = 0$$

Iteration	α	β
0	10.0000	20.0000
1	16.0000	65.6000
2	10.7049	26.8887
3	13.3716	45.3821
4	11.3410	31.5853
5	12.5141	39.3944
6	11.7131	34.1273
7	12.1931	37.2498
8	11.8865	35.2711
9	12.9713	36.4553
10	11.9575	35.7305
11	12.0257	36.1631
12	11.9846	35.9034
13	12.0091	36.0575
14	11.9946	35.9661
15	12.0032	36.0199
16	11.9981	35.9883
17	12.0011	36.0068
18	11.9993	35.9950
19	12.0003	36.0023
20	11.9997	35.9986
21	12.0001	36.0008
22	11.9999	35.9995
23	12.0000	36.0002
24	11.9999	35.9998
25	12.0000	36.0001
26	12.0000	35.9999
27	12.0000	36.0000
28	12.0000	36.0000

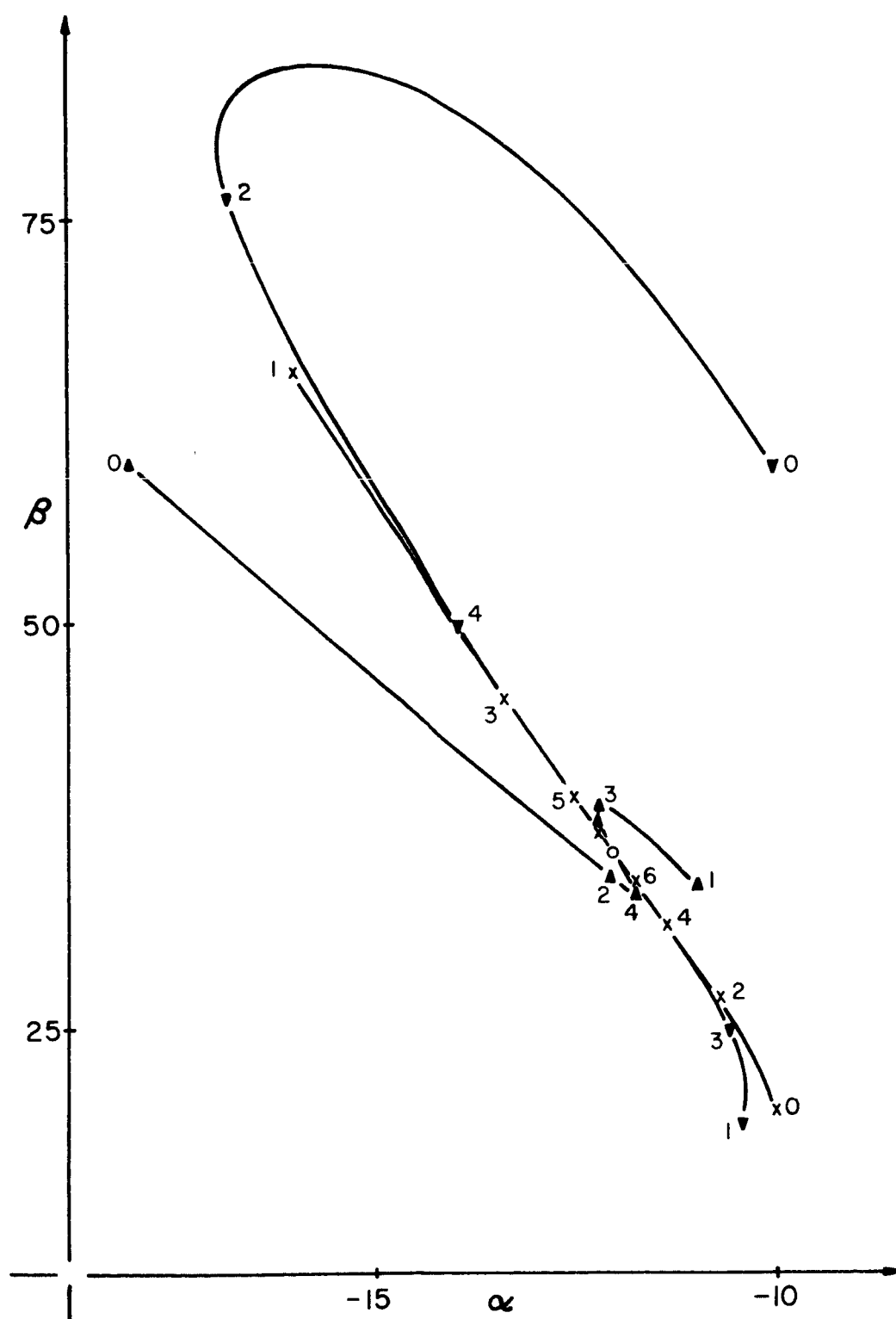


Figure 11

Successive Points of the Iteration for the Quartic

$$z^4 - 10z^3 + 20z^2 - 24z + 288 = 0$$

Consider the equation

$$z^4 - 10.4z^2 - 19.2z - 12.8 = 0 \quad (110)$$

$$(z^2 - 2z - 8) (z^2 + 2z + 1.6) = 0 \quad (111)$$

$$(z + 2) (z - 4) (z + 1 + j 0.774) (z + 1 - j 0.774) = 0 \quad (112)$$

It is found for the λ 's

$$\lambda_{-1 \pm j 0.774} = \pm j 4.12 \quad (113)$$

$$\lambda_{-2} = 0.6 \quad ; \quad \lambda_4 = -0.6 \quad (114)$$

The iteration converges therefore toward the point -2, -8 of the α, β -plane on two curves depending upon the initial trial factor and with a rate of convergence determined by $|\lambda_{-2}| = |\lambda_4|$.

The results of the iteration are shown in Table 3 for the case in which the initial factor is 0, -10.4. The corresponding curve and curves for other initial values are shown in Figure 12.

It can be seen from Figure 12, that there are no common tangents to the curve at the point -2, -8 as could be expected from the theory considered earlier.

Example 4: Complex conjugate characteristic roots
Consider the quintic

Table 3

Results of the Iteration for the Quartic

$$z^4 - 10.4z^2 - 19.2z - 12.8 = 0$$

Iteration	α	β
0	0.00000	-10.4000
1	-1.84615	-11.6307
2	-1.47610	-8.77541
3	-1.94258	-8.99117
4	-1.82785	-8.27288
5	-1.97898	-8.32994
6	-1.93987	-8.09764
7	-1.99238	-8.11573
8	-1.97858	-8.03509
9	-1.99725	-8.04129
10	-1.99232	-8.01262
11	-1.99901	-8.01482
12	-1.99724	-8.00455
13	-1.99964	-8.00533
14	-1.99900	-8.00164
15	-1.99987	-8.00192
16	-1.99964	-8.00059
17	-1.99995	-8.00069
18	-1.99987	-8.00021
19	-1.99998	-8.00025
20	-1.99995	-8.00008
21	-1.99999	-8.00009
22	-1.99998	-8.00003
23	-1.99999	-8.00003
24	-1.99999	-8.00001
25	-2.00000	-8.00001
26	-1.99999	-8.00001
27	-2.00000	-8.00001
28	-2.00000	-8.00000

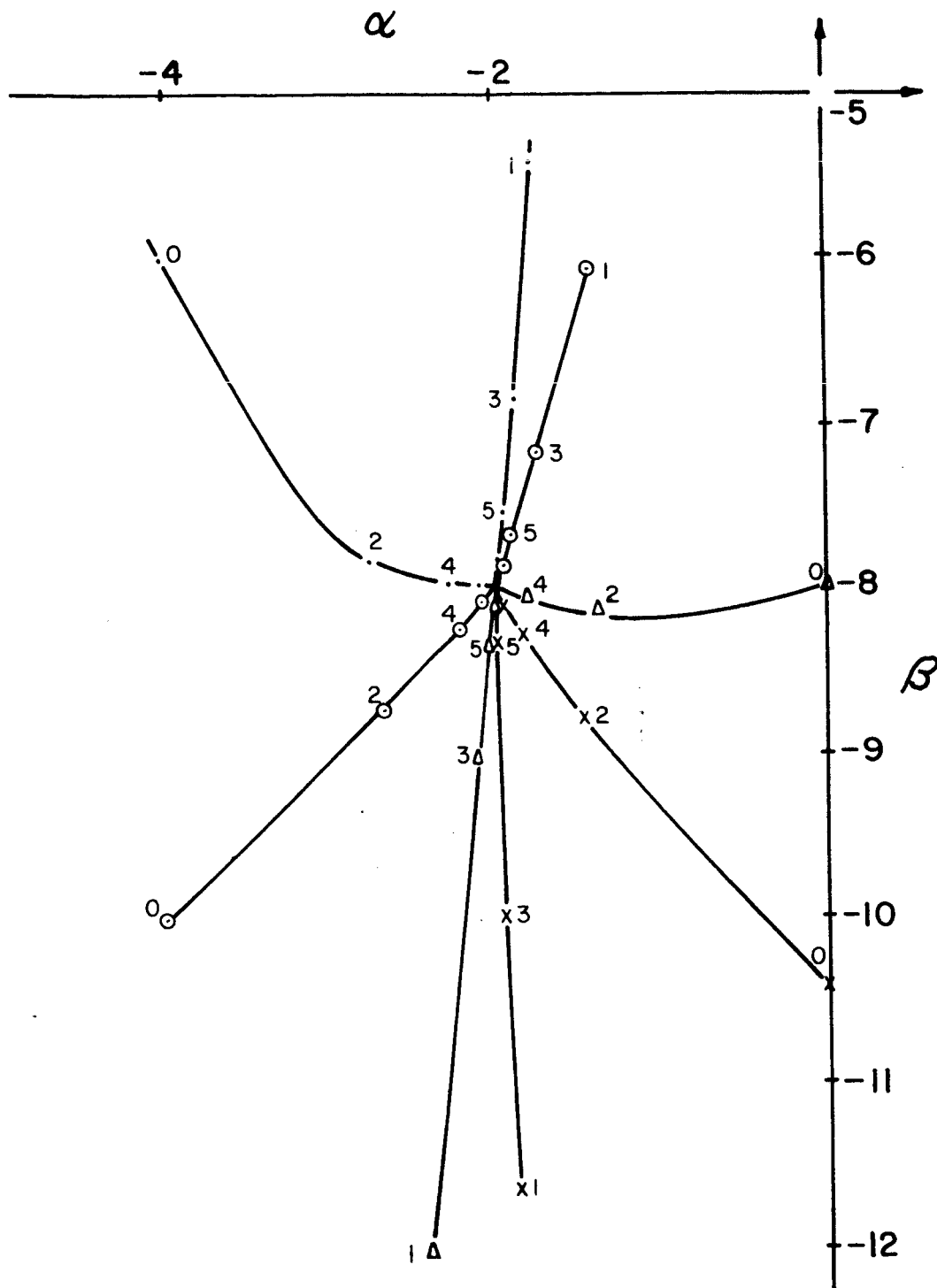


Figure 12

Successive Points of the Iteration for the Quartic
 $z^4 - 10.4z^2 - 19.2z - 12.8 = 0$, for Different Initial Trial Factors

$$z^5 + 4z^4 + 11z^3 + 15z^2 + 12z + 5 = 0 \quad (115)$$

$$(z^2 + 2z + 5)(z^2 + z + 1)(z + 1) = 0 \quad (116)$$

$$(z + 1 + 2j)(z + 1 - 2j)(z + 0.5 + j 0.866) \\ (z + 0.5 - j 0.866)(z + 1) = 0 \quad (117)$$

The λ 's are found to be

$$\lambda_{-1 \pm 2j} = \frac{69 \pm 58j}{125} ; \quad \lambda_{-1 \pm 2j} = 0.721 e^{\pm j \frac{40\pi}{180}} \quad (118)$$

$$\lambda_{-0.5 \pm j 0.866} = \mp 3.464j ; \quad \lambda_{-0.5 \pm j 0.866} = 3.464 e^{\mp j \frac{\pi}{2}} \quad (119)$$

The iteration converges toward the point 2, 5 of the α, β -plane with $\rho = 0.721$ and $\theta = 40^\circ$ which means that approximately 9 iterations are necessary to describe a portion of the curve corresponding to an angle of 2π , and that for such a rotation the error would decrease in a ratio

$$(0.721)^9 \simeq 0.05 \quad (120)$$

The results are given in Table 4 and the corresponding curve is plotted in Figure 13.

From the study and the examples given in this section, it is possible to see that the convergence of the iteration can only follow a certain number of definite patterns. The next problem which arises is to see how this knowledge can be used to speed up the factorization of the given polynomials.

Table 4
Results of the Iteration for the Quintic

$$z^5 + 4z^4 + 11z^3 + 15z^2 + 12z + 5 = 0$$

Iteration	α	β
0	4.0000	11.0000
1	3.0143	6.1315
2	2.4517	4.7766
3	2.0925	4.3486
4	1.8766	4.3506
5	1.7923	4.5944
6	1.8231	4.9150
7	1.9250	5.1424
8	2.0083	5.2028
9	2.0488	5.1458
10	2.0483	5.0562
11	2.0281	4.9883
12	2.0063	4.9584
13	1.9923	4.9600
14	1.9882	4.9774
15	1.9909	4.9959
16	1.9962	5.0073
17	2.0005	5.0102
18	2.0025	5.0075
19	2.0025	5.0029
20	2.0015	4.9993
21	2.0003	4.9978
22	1.9996	4.9979
23	1.9994	4.9988
24	1.9995	4.9998
25	1.9998	5.0004
26	2.0000	5.0005
27	2.0001	5.0004
28	2.0001	5.0002
29	2.0001	5.0000
30	2.0000	4.9999

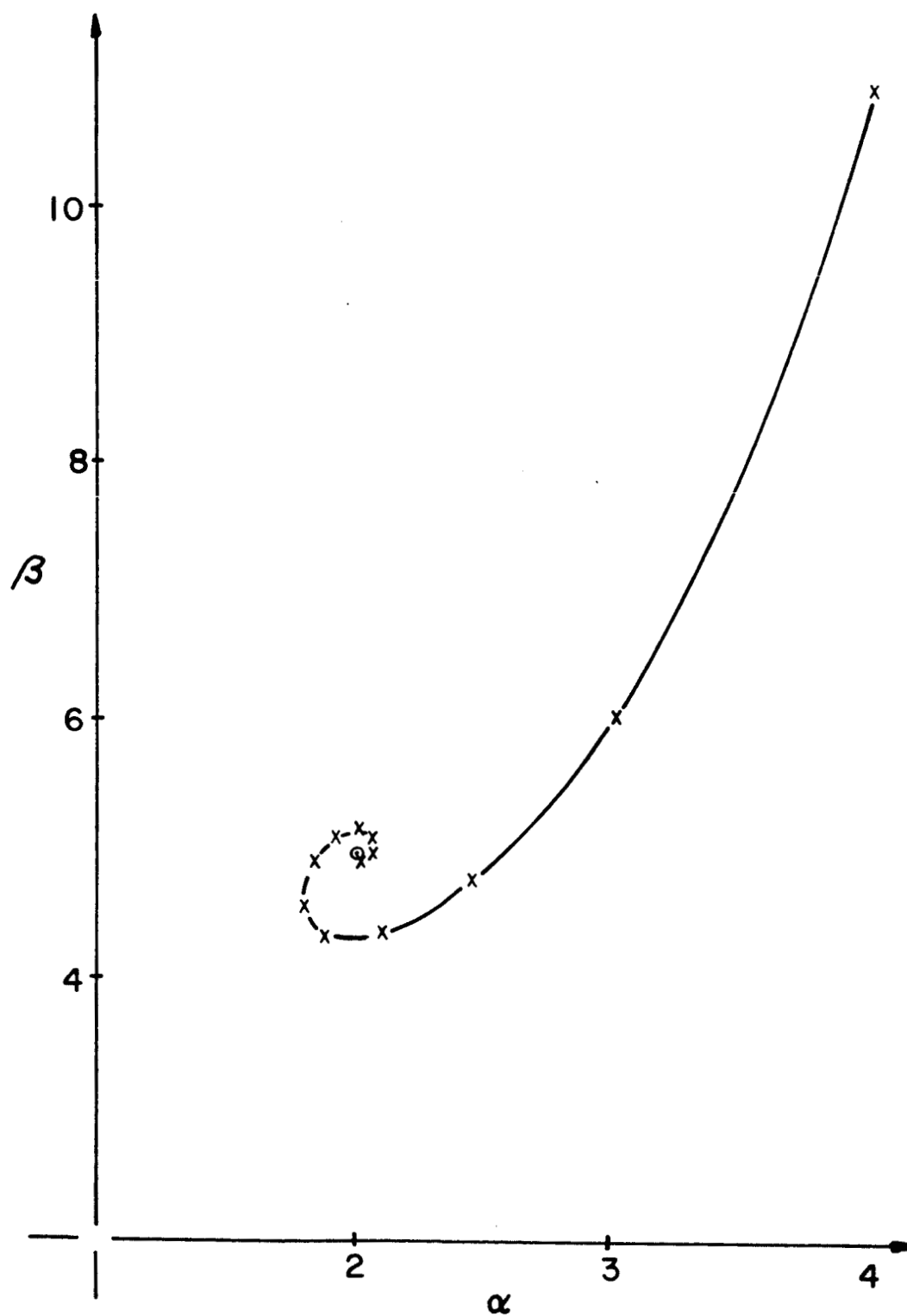


Figure 13

Successive Points of the Iteration for the Quintic
 $z^5 + 4z^4 + 11z^3 + 15z^2 + 12z + 5 = 0$, for Different Initial Trial Factors

3. UTILIZATION OF THE PREVIOUS RESULTS TO SPEED UP THE FINDING OF ROOTS

In the previous section the convergence of the iteration process was studied and interpreted in the α, β -plane. It was shown that depending upon the nature, the sign and the magnitudes of the characteristic roots of the matrix corresponding to a linearization, different types of convergence could occur. A few examples were shown to illustrate these results.

In practice, however, it is interesting to find the roots as fast as possible and with as good an accuracy as possible, and it is always desirable to speed up the solution and to reduce the number of calculations as much as possible. Knowing from the previous chapter the different patterns which the successive iterations may follow, it would be possible once the characteristic roots of the matrix corresponding to any specific iteration process are known to improve on the accuracy of the coefficients of a quadratic factor by using an interpolation or an extrapolation.

This seems paradoxical since the values of the λ 's are required and these can be calculated only once the searched value is found, which leads to the conclusion that if the answer is known the problem can be solved. However, the

knowledge of the characteristic roots of M is not necessary, because with very little or even without any previous experience a user of the method can recognize the class of convergence to which any specific iteration belongs after just a few iterations. Then with the use of a numerical or graphical interpolation (or extrapolation) values of the coefficients of the quadratic factor much closer to the exact values can be found. Repeating a few times the process of combining iterations and interpolations (or extrapolations) the accuracy on the coefficients can be improved as much as desired.

This method is very powerful, because with very little extra effort, once the type of convergence has been found, its user may improve a lot on the accuracy and get as many significant digits as he wishes, without having to carry many superfluous ones in his calculations.

In the following examples some problems similar to those which might arise in practice are considered.

Example 5:

Consider the quartic

$$z^4 + 20z^3 + 130z^2 + 370z + 400 = 0 \quad (121)$$

After four iterations, it can be seen that the considered case corresponds to either a case of both λ 's positive or both complex conjugate with a very low value of θ . A continuation of the iteration process would yield more precision

about the nature of the λ 's but this is not necessary.

Extrapolating the results which might be seen in the first set of iterations of Table 5, or on the curve shown in Figure 14, it can be seen that 14.0, 35 should be a good next approximation.

Iterating three times again suggests 13.9, 33.1 as being closer to the final value, and repeating the process once more would yield 13.892, 33.04.

Finally the quadratic factor, its coefficients defined with 6 significant digits, is found to be

$$p^*(z) = z^2 + 13.8913z + 33.0332 \quad (122)$$

And the given quartic of Equation (121) can be factorized

$$(z^2 + 13.8913z + 33.0332)(z^2 + 6.1087z + 12.1090) = 0 \quad (123)$$

The results of the successive iterations are shown in Table 5, and a part of these results has been illustrated in Figure 14.

It is interesting to notice how a relatively small amount of calculations has led to a very high accuracy.

Example 6:

Consider the quartic

$$z^4 + 9z^3 - 100z^2 - 950z - 2040 = 0 \quad (124)$$

Five or six iterations are sufficient to recognize that this is the case of two λ 's of opposite signs with the negative

Table 5

Results of Successive Iterations for the Quartic

$$z^4 + 20z^3 + 130z^2 + 370z + 400 = 0$$

First Set of Iterations

Iteration	0	1	2	3	4
α	20.0	17.6	16.5	15.8	15.3
β	130	79.5	62.6	54.0	48.6

Second Set of Iterations

Iteration	0	1	2	3
α	14.00	14.00	13.98	13.96
β	35.00	34.57	34.19	33.89

Third Set of Iterations

Iteration	0	1	2	3
α	13.900	13.896	13.895	13.893
β	33.100	33.077	33.063	33.054

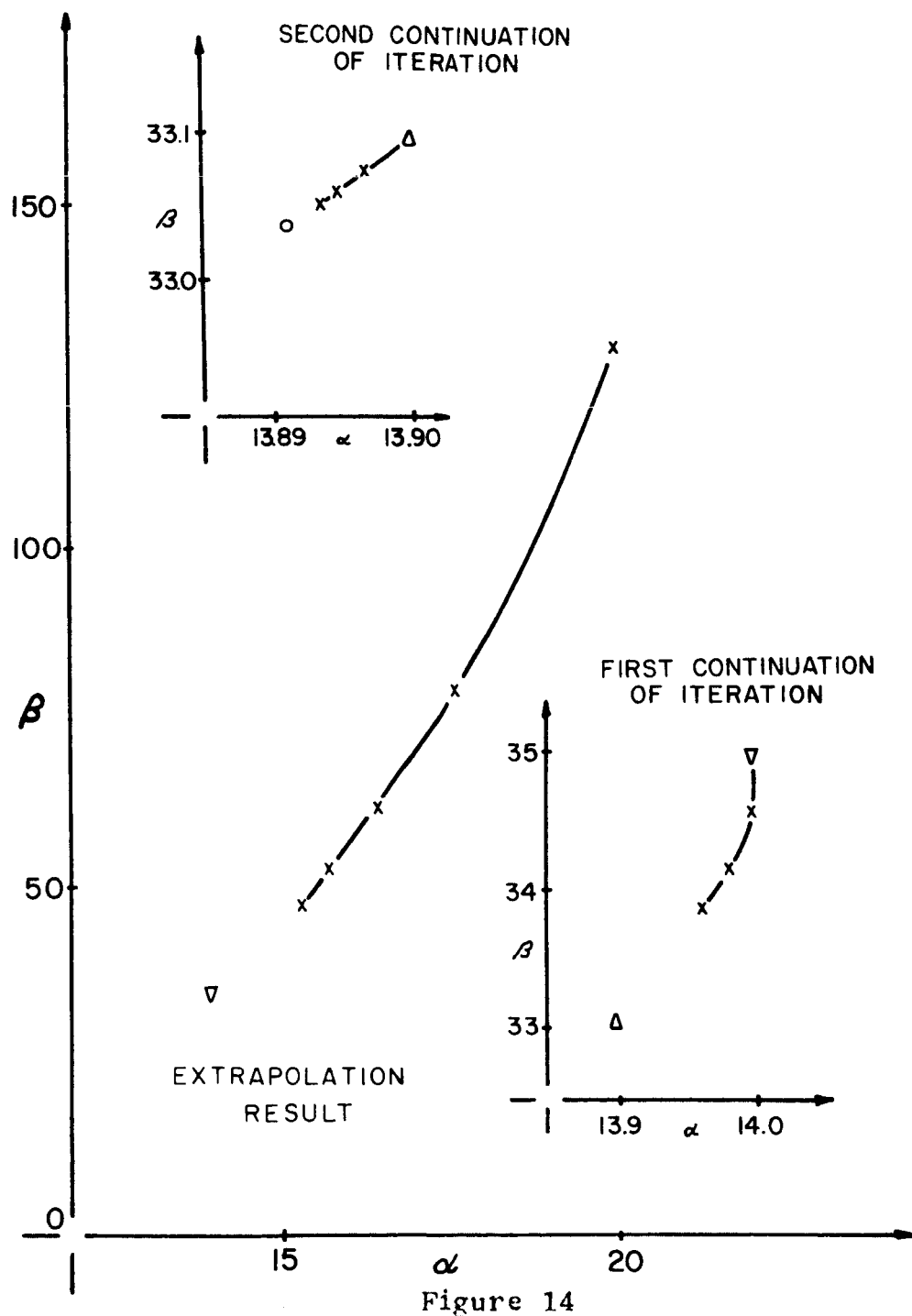


Figure 14
Points of Successive Iterations for the Determination of the
Roots of the Quartic $z^4 + 20z^3 + 130z^2 + 370z + 400 = 0$

having the larger absolute value. By interpolation a good approximation to the exact coefficients of the quadratic factor can be obtained. If more accuracy is desired, the iteration may be continued from these values, the knowledge of the type of convergence obtained in the first set of iterations allowing a reduction in the number of iterations necessary to obtain the next approximation. The accuracy with which the quadratic factor is obtained increases with the number of times the combination "iterations plus interpolation" is repeated.

For the case of Equation (124), it can be seen from Table 6, that after eight iterations a rough interpolation gives as first approximation to the quadratic factor $p(z)$:

$$p(z) = z^2 + 1.30z - 125.0 \quad (125)$$

Taking these values for next trial factor and iterating six times as shown in the second part of Table 6, the factor becomes after interpolation:

$$p(z) = z^2 + 1.305z - 126.23 \quad (126)$$

Repeating the same process again the correct value is obtained within six significant digits, and the quartic of Equation (124) can be factorized into the expression:

$$(z^2 + 1.30562z - 126.209)(z^2 + 7.69438z + 16.1635) = 0 \quad (127)$$

The curve illustrating the general behaviour of the

Table 6

Results of Successive Iterations for the Quartic $z^4 + 9z^3 - 100z^2 - 950z - 2040 = 0$

First Set of Iterations

Iteration	0	1	2	3	4	5	6	7	8
α	9.0	-2.3	4.8	-1.6	3.6	-0.9	2.9	-0.3	2.5
β	-100	-222	-99.4	-171	-103	-154	-108	-146	-112

Second Set of Iterations

Iteration	0	1	2	3	4	5	6
α	1.300	1.230	1.328	1.263	1.331	1.277	1.326
β	-125.0	-126.4	-125.6	-126.5	-125.8	-126.5	-125.9

Third Set of Iterations

Iteration	0	1	2	3	4	5
α	1.30500	1.30698	1.30488	1.30650	1.30498	1.30625
β	-126.230	-126.200	-126.222	-126.201	-126.218	-126.202

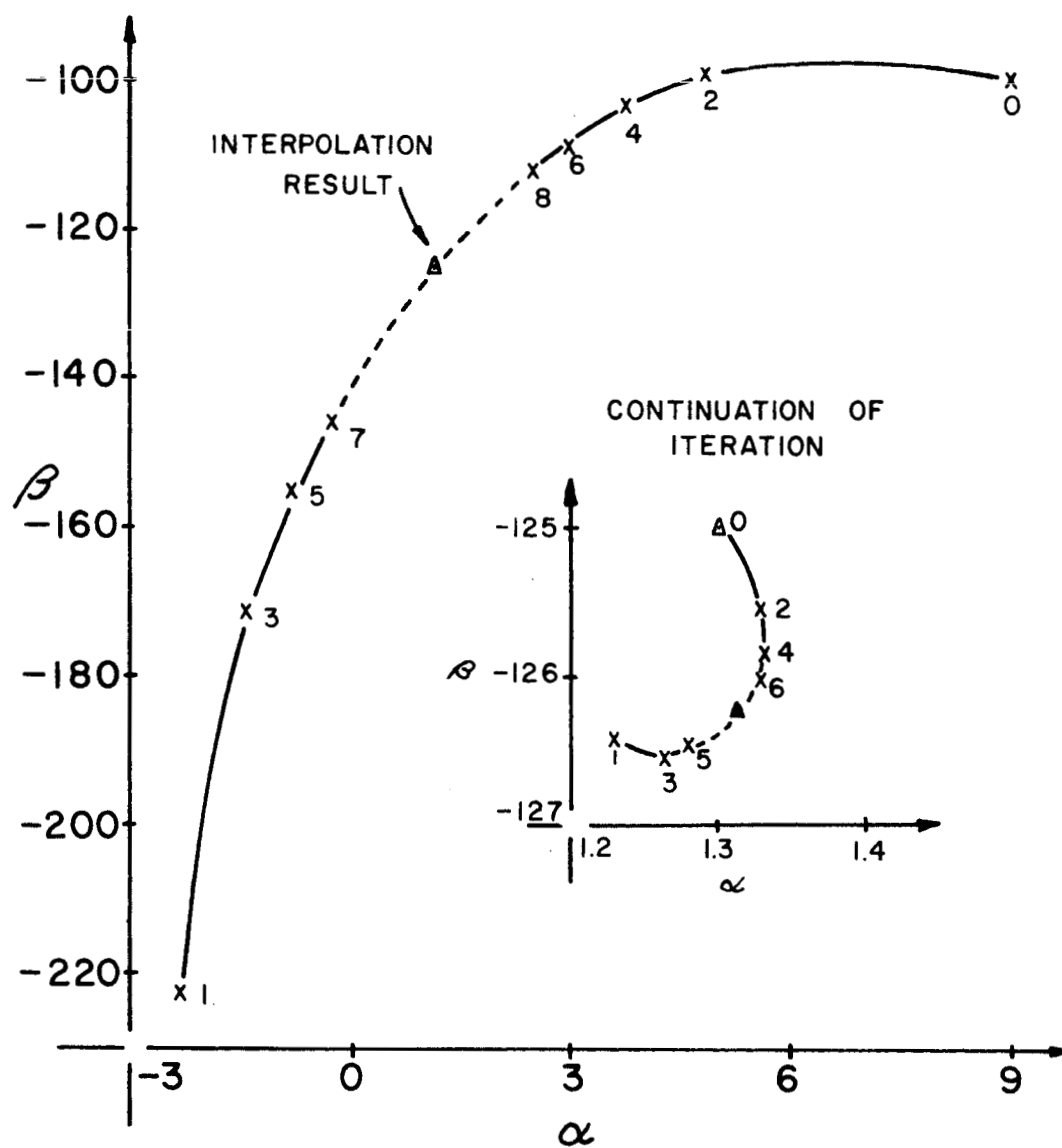


Figure 15

Points of Successive Iterations for the Determination of the
 Roots of the Quartic $z^4 + 9z^3 - 100z^2 - 950z - 2040 = 0$

convergence in the α, β -plane is given in Figure 15.

Example 7:

Consider the quintic

$$z^5 + 4.5z^4 + 9z^3 + 9z^2 + 4.7z + 1.5 = 0 \quad (128)$$

From the first few iterations one can expect either the case of two positive λ 's corresponding to two real roots, or two complex conjugate λ 's with small values of θ . A few more iterations would give more information on the nature of the λ 's but this is not necessary in order to find the quadratic factor. After three iterations an extrapolation is made which yields for next trial factor:

$$p(z) = z^2 + 2.5z + 2.5 \quad (129)$$

The second set of iterations (also shown in Table 7) shows that the direction of the points of the iteration has changed which means that the λ 's are conjugate complex. From there on, remembering that the searched values have to correspond to a point located inside the concavity of the curve in the α, β -plane, an approximation as good as desired to $p^*(z)$ can be obtained very quickly as shown in Table 7. In the case of the quintic of Equation (128) the coefficients of $p^*(z)$ obtained within six significant digits allow the quintic to be factorized as:

$$(z^2 + 2.64241z + 2.95485)$$

$$(z^3 + 1.85758z^2 + 1.13664z + 0.507640) = 0 \quad (130)$$

Table 7

Results of Successive Iterations for the Quintic

$$z^5 + 4.5z^4 + 9z^3 + 9z^2 + 4.7z + 1.5 = 0$$

First Set of Iterations

Iteration	0	1	2	3
α	4.5	3.7	3.3	3.0
β	9.0	5.1	3.9	3.2

Second Set of Iterations

Iteration	0	1	2	3
α	2.5	2.42	2.41	2.46
β	2.5	2.52	2.64	2.82

Third Set of Iterations

Iteration	0	1	2	3
α	2.60	2.65	2.68	2.69
β	3.00	3.05	3.06	3.04

Fourth Set of Iterations

Iteration	0	1	2	3
α	2.6400	2.6400	2.6407	2.6416
β	2.9500	2.9516	2.9535	2.9551

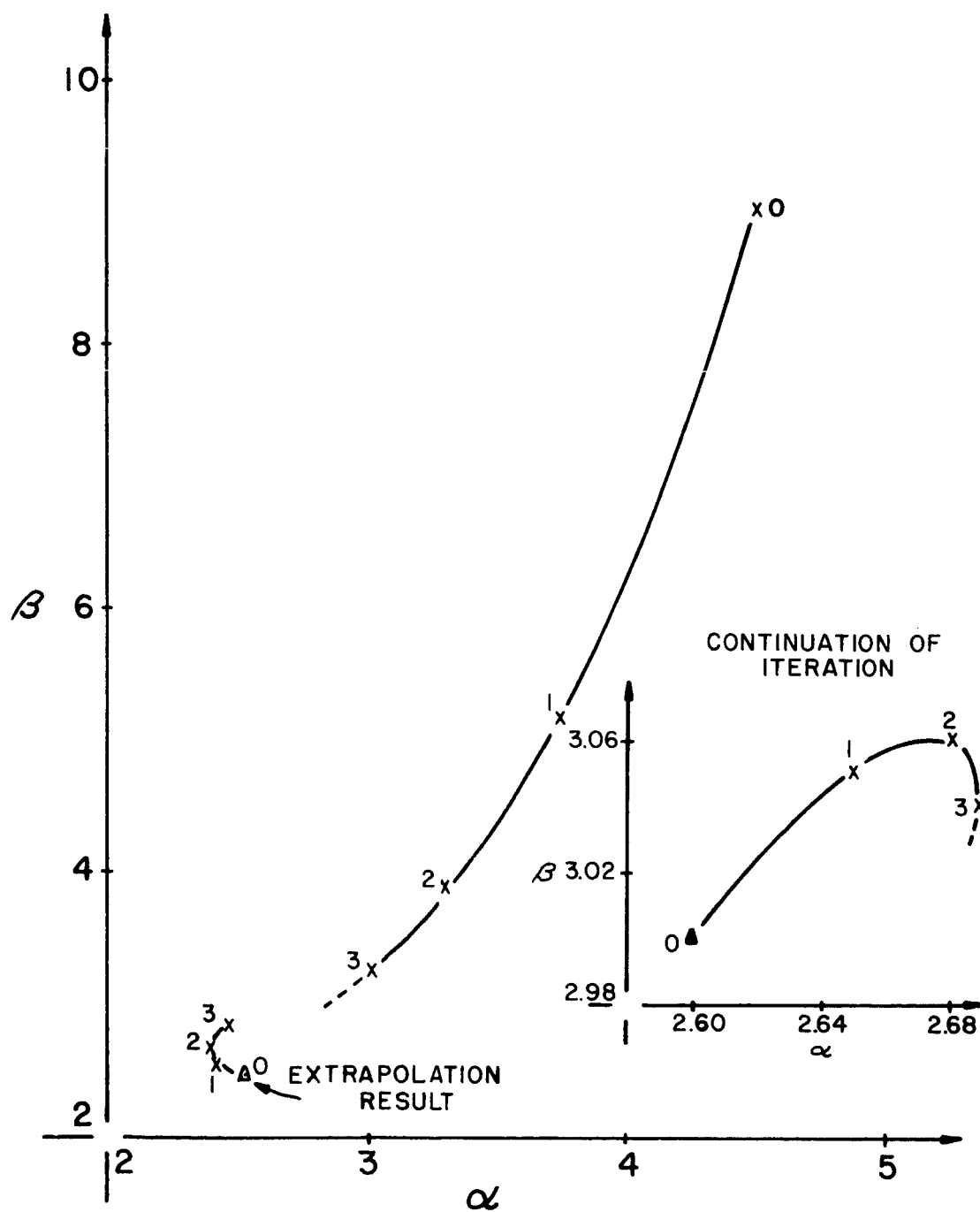


Figure 16

Points of Successive Iterations for the Determination of the
 Roots of the Quintic $z^5 + 4.5z^4 + 9z^3 + 9z^2 + 4.7z + 1.5 = 0$

The curve representing the progress of the iteration is given in Figure 16.

Example 8:

Consider the sextic:

$$z^6 + 14z^5 + 77z^4 + 265z^3 + 700z^2 + 1940z + 3400 = 0 \quad (131)$$

In this particular case a new phenomenon can be observed due to the fact that the linear approximation is valid only in a close neighborhood of the final value. From the first three results of the iteration, a convergence corresponding to two positive λ 's or two complex conjugate λ 's could be expected. However after the third iteration there is a sudden jump easily noticeable in Figure 17. Carrying the iteration further, it becomes apparent that the two λ 's are in fact real and negative. From there on, by combining Oldenburger's right to left synthetic division and a simple interpolation process, one can find as good an approximation as is desired to the quadratic factor. In Table 8 the successive results of the iteration are shown and the corresponding curves are plotted in Figure 17. Knowing the coefficients of the quadratic factor within six digits, the sextic of Equation (131) can be factorized:

$$(z^2 + 10.2829z + 23.2184)$$

$$(z^4 + 3.7171z^3 + 15.5592z^2 + 18.7012z + 146.435) = 0 \quad (132)$$

To finish solving the problem, the roots of the quartic

Table 8

Results of Successive Iterations for the Sctic $z^6 + 14z^5 + 77z^4 + 265z^3 + 700z^2 + 1940z + 3400 = 0$

First Set of Iterations

Iteration	0	1	2	3	4	5	6	7
α	14.00	11.76	10.46	9.81	10.56	10.10	10.51	10.53
β	77	40	27	20.9	25.25	21.94	24.78	22.18

Second Set of Iterations

Iteration	0	1	2	3	4
α	10.290	10.266	10.297	10.270	10.296
β	23.300	23.108	23.317	23.126	23.317

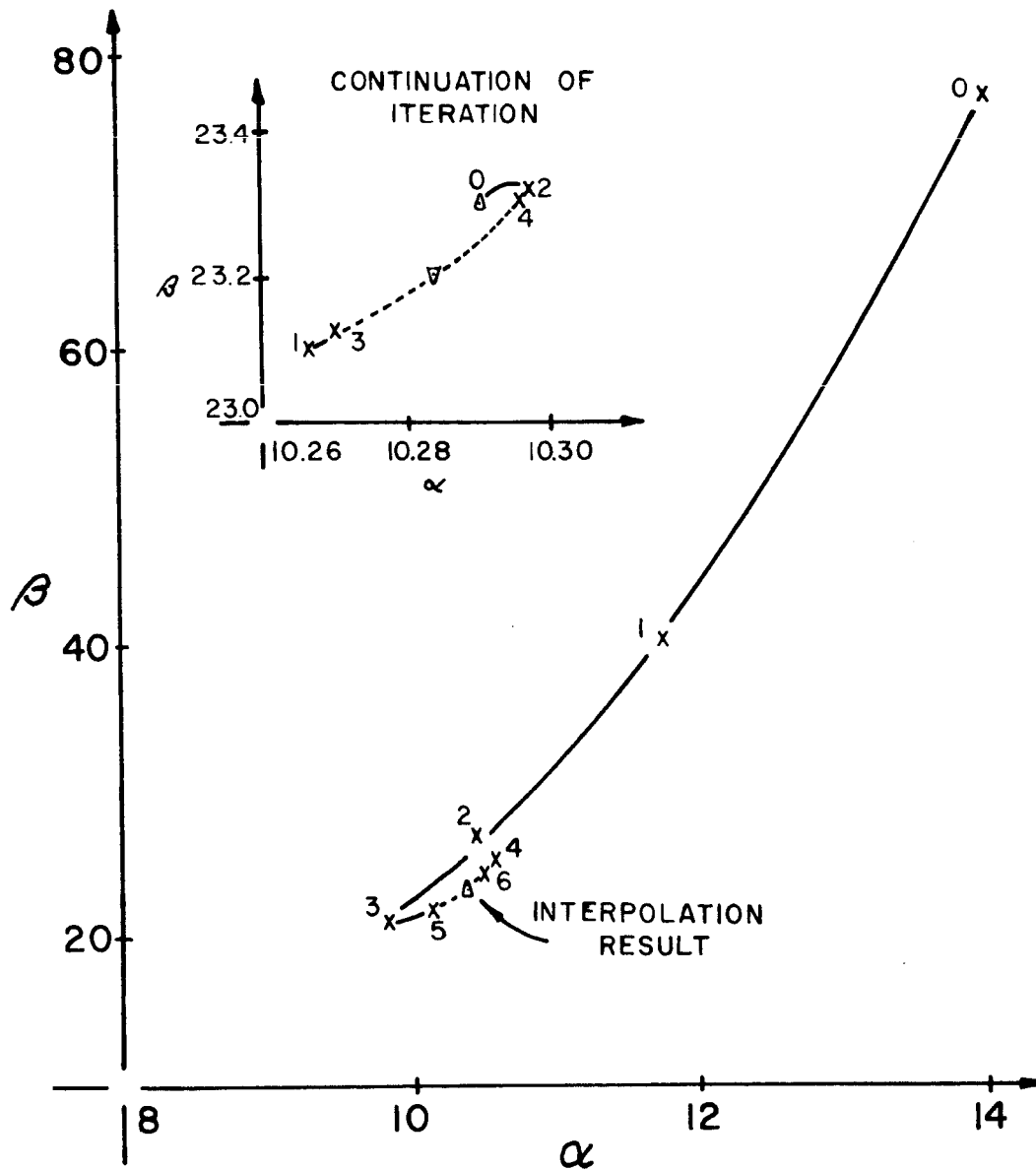


Figure 17

Points of Successive Iterations for the Determination
of the Roots of the Sextic

$$z^6 + 14z^5 + 77z^4 + 265z^3 + 700z^2 + 1940z + 3400 = 0$$

must be determined. With four iterations the case of two complex conjugate λ 's with an argument of approximately 100° is recognized. From thereon, for each new trial value one or two iterations are sufficient to indicate a better approximation since the searched value is known to lie inside the concavity of the curves.

The results of the iterations are given in Table 9, and the corresponding curves plotted in Figure 18. When the quadratic factor has been found with six significant digits, the sextic of Equation (131) can be written as:

$$\begin{aligned} (z^2 + 10.2829z + 23.2185) (z^2 + 5.50967z + 16.6305) \\ (z^2 - 1.79257z + 8.80514) = 0 \end{aligned} \quad (133)$$

Other examples of converging iterations could be considered to illustrate more cases of problems for which the iteration converges, but it is believed that this would be superfluous at this point to consider any more cases for which the characteristic roots of the matrix corresponding to the linear approximation are both smaller than one in absolute value, since the problems which would be encountered would have the same nature.

Table 9

Results of Successive Iterations for the Quartic

$$z^4 + 3.7171z^3 + 15.5592z^2 + 18.7012z + 146.435 = 0$$

First Set of Iterations

Iteration	0	1	2	3	4
α	3.7	4.9	8.9	4.9	4.5
β	15	10	26	20	12

Second Set of Iterations

Iteration	0	1	2
α	6.00	5.38	5.23
β	18.00	17.46	15.35

Third Set of Iterations

Iteration	0	1	2
α	5.500	5.513	5.516
β	16.60	16.61	16.66

Fourth Set of Iterations

Iteration	0	1	2
α	5.5100	5.5101	5.5093
β	16.630	16.633	16.630

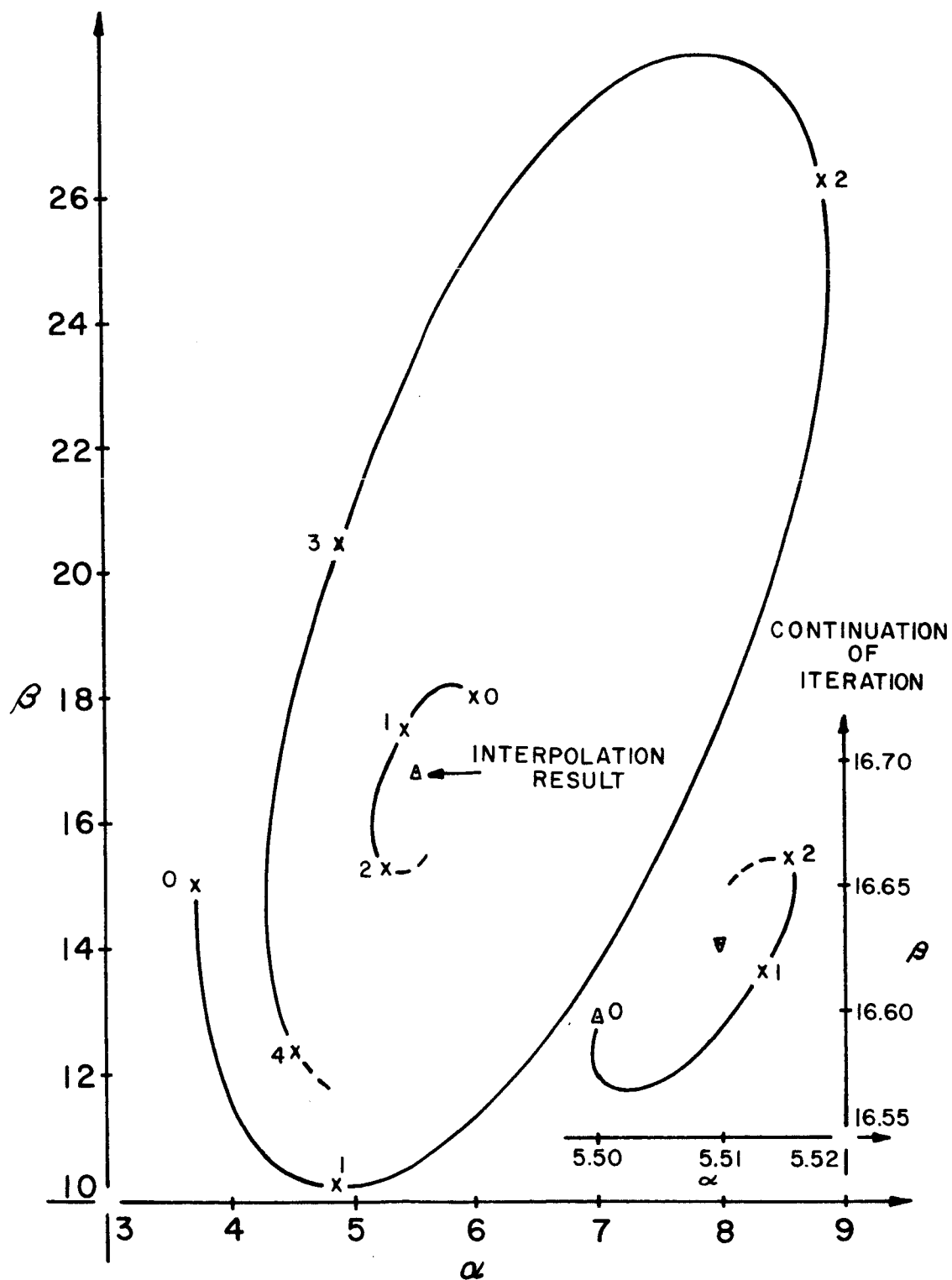


Figure 18

Points of Successive Iterations for the Determination of the Roots
of the Quartic $z^4 + 3.7171z^3 + 15.5592z^2 + 18.7012z + 146.435 = 0$

4. EXTENSION OF THE METHOD TO DIVERGING ITERATIONS

In the previous chapters only equations for which the method converged were considered. Although this covers most equations, the method can be extended, whenever necessary, to some diverging iterations.

It will be shown that some of the properties found for converging iterations apply also to diverging cases. However, the diverging process is not well suited for making any general study because as the iteration progresses the points move out of the region where the linear approximation is valid. The methods considered below are therefore more fitted for increasing the accuracy of a set of coefficients than to determine them in the first place, especially when the quadratic factor corresponds to two real λ 's.

The phenomena which occur are in nature similar to those found when the λ 's are smaller than one in absolute value. They can be now classified in four general categories which are considered next.

Complex Conjugate λ 's with Moduli Larger than One

For both λ 's complex conjugate and greater than one in absolute value, the successive points of the iteration lie on the curve spiralling around and outwards from the searched value. The locus of the points of the iteration has the same

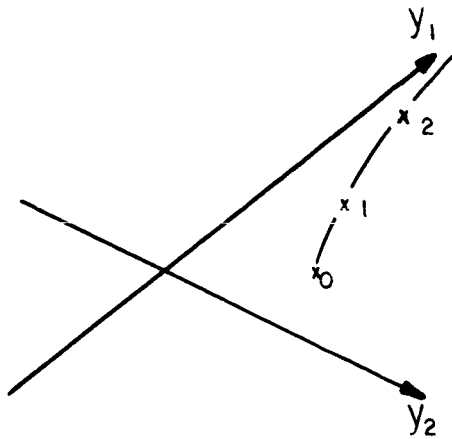
aspect as that found for the converging case and that was illustrated in Figure 9. The rate of increase of the error-vector and the angle of rotation are determined as was done before. The only main difference is that in the case of $|\lambda| > 1$ the successive error vectors increase instead of decreasing.

$$\text{Real } \lambda \text{'s, } |\lambda_1| > 1 > |\lambda_2|$$

For the case of two real λ 's one of which is larger than one in absolute value, the other one smaller, it can be seen from Equation (71) which is still valid, that one of the components of Y_k increases while the other one decreases for successive steps of the iteration. The points of the iteration diverge therefore from the searched value asymptotically to the axis corresponding to $|\lambda| > 1$. The same study which was done for $|\lambda_1| \neq |\lambda_2| < 1$ could be made again but since the reasoning is exactly the same, the graphical representation of the four possible cases which is given in Figure 19 should be sufficient to illustrate the different possibilities.

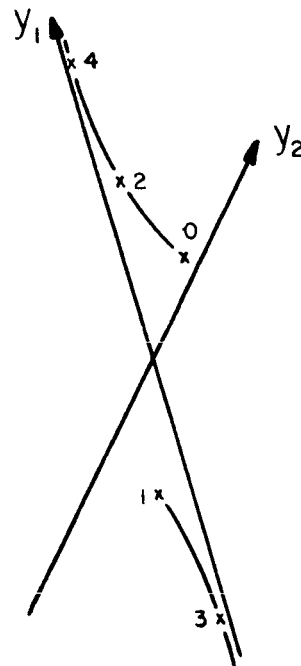
$$\text{Real } \lambda \text{'s, } |\lambda_i| > 1, i = 1, 2 \text{ but } m_{21} \neq 0, \frac{A}{B} \neq \frac{2m_{11}}{m_{21}}$$

For the case of two real λ 's both greater than one in absolute value, the two components of the vector Y_k diverge. Therefore there are no asymptotes, but there is an asymptotic direction which is that of the axis corresponding to the greater λ in absolute value. Here again four cases are possible depending upon the signs and magnitudes of the λ 's



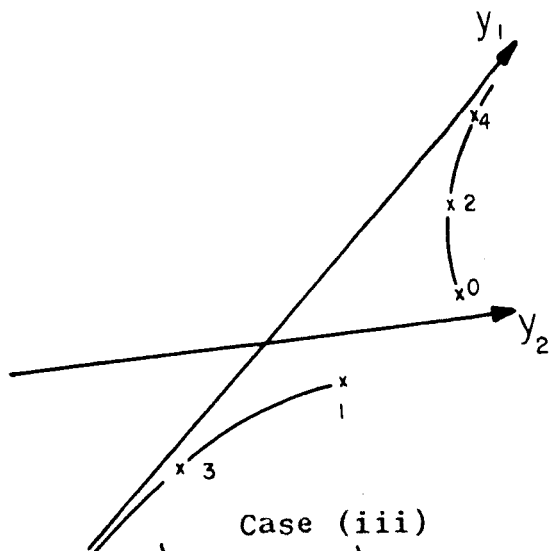
Case (i)

$$\lambda_1 > 1 ; 1 > \lambda_2 > 0$$



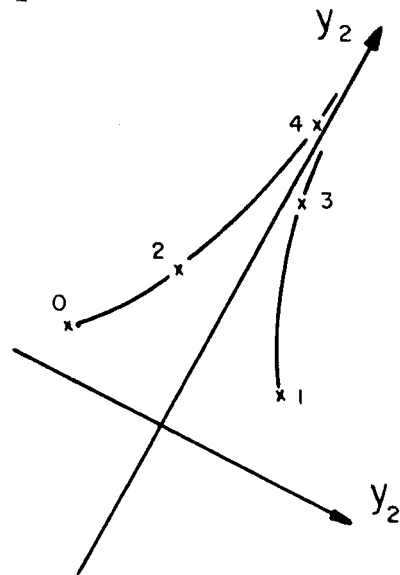
Case (ii)

$$\lambda_1 < -1 ; 0 < \lambda_2 < -1$$



Case (iii)

$$\lambda_1 < -1 ; 1 > \lambda_2 > 0$$



Case (iv)

$$\lambda_1 > 1 ; -1 < \lambda_2 < 0$$

Figure 19

Divergence Patterns for the Case of
Real Characteristic Roots, $|\lambda_1| > 1 > |\lambda_2|$

and they have been illustrated in Figure 20.

Real λ 's of Equal Magnitudes Larger than One in

$$\text{Absolute Value with } m_{21} = 0 \text{ or } \frac{A}{B} = \frac{2m_{11}}{m_{21}}$$

When $m_{21} = 0$ or $\frac{A}{B} = \frac{2m_{11}}{m_{21}}$ and the λ 's are greater than one in absolute value, the points of the iteration diverge from the searched value on one or two lines depending upon the considered case. The graphical representations of that case would be similar to those seen in Figure 8, where however the successive points of the iteration diverge from the searched value instead of converging to it.

It can therefore be seen from the curves shown in Figures 19 and 20, as well as from the curves given in Figures 8 and 9 (for which the order of the successive iterations has been inverted) that by successive approximations it is still possible to determine the quadratic factor although the iteration diverges.

One should not minimize however the fact that the applicability of these results is reduced because the iteration has a tendency to take the successive points out of the domain in which linearity can be reasonably assumed. The choice of a good first trial factor in the case of diverging iterations becomes of paramount importance.

Two examples are going to be considered next to illustrate how the method can still be used in some cases of

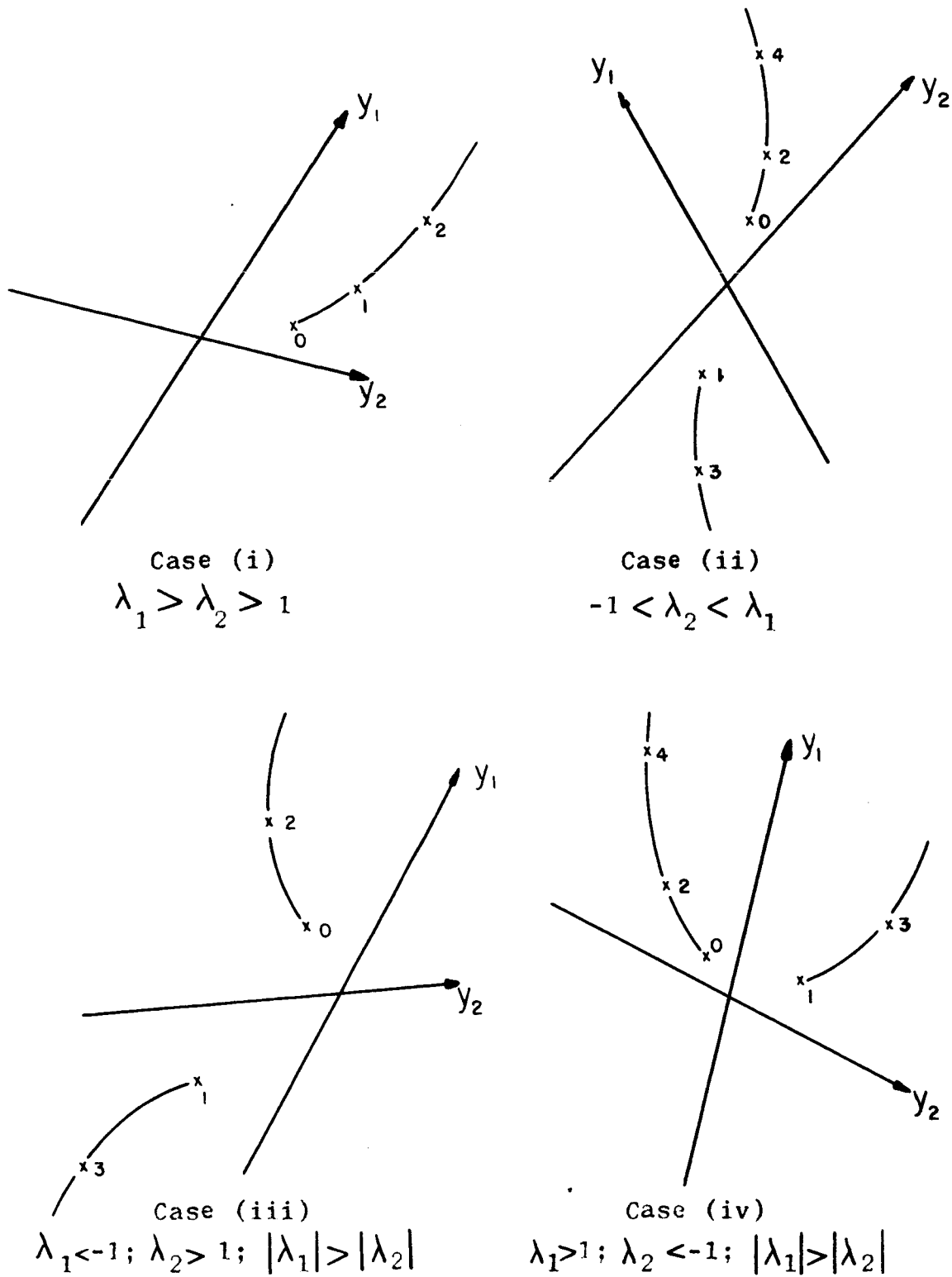


Figure 20

Divergence Patterns for the Case of
 Real Characteristic Roots $|\lambda_1| > |\lambda_2| > 1$

diverging iterations.

Example 9:

Consider the quartic

$$z^4 - 2z^3 + 27z^2 - 61z + 291 = 0 \quad (134)$$

Five or six iterations are sufficient to show that the iteration does not converge although the successive values of the iteration indicate the presence of a possible solution in the neighborhood of the point 4, 25. Taking these values for the next trial factor, it may be seen that the iteration diverges with an argument $\theta = 80^\circ$. Using the properties concerning the concavity of the curves and the position of the searched point, one may approximate the exact value by successive trials until the desired accuracy is obtained. The method is illustrated for the case of Equation (134) in Table 10 and Figure 21. By successive repetition of this method, the quadratic factor was found within 6 significant digits and the quartic of Equation (134) factorized as

$$(z^2 + 2.02786z + 21.8497)(z^2 - 4.02786z + 13.3182) = 0 \quad (135)$$

Example 10:

Consider the quartic

$$z^4 + 7z^3 - 64z^2 - 592z - 1220 = 0 \quad (136)$$

A few iterations are sufficient to realize that the method diverges, and that the considered case is that of two

Table 10

Results of Successive Iterations for the Quartic $z^4 - 2z^3 + 27z^2 - 61z + 291 = 0$

First Set of Iterations						
Iteration	0	1	2	3	4	5
α	-2	-0.5	1.7	68	6.2	-1.9
β	27	13	3.1	53	579	27
						13.5
Second Set of Iterations						
Iteration	0	1	2	3	4	5
α	4.0	2.3	0.5	0.8	2.7	7.4
β	25.0	32.6	23.8	16.2	12.9	29.6
						50.9
						25.1
						14.3
						7.5
Third Set of Iterations						
Iteration	0	1	2	3	4	5
α	2.000	1.975	2.026	2.098	2.082	
β	22.00	21.72	21.56	21.80	22.22	
Fourth Set of Iterations						
Iteration	0	1	2	3	4	5
α	2.020	2.035	2.044	2.030	2.008	
β	21.80	21.80	21.88	21.94	21.87	

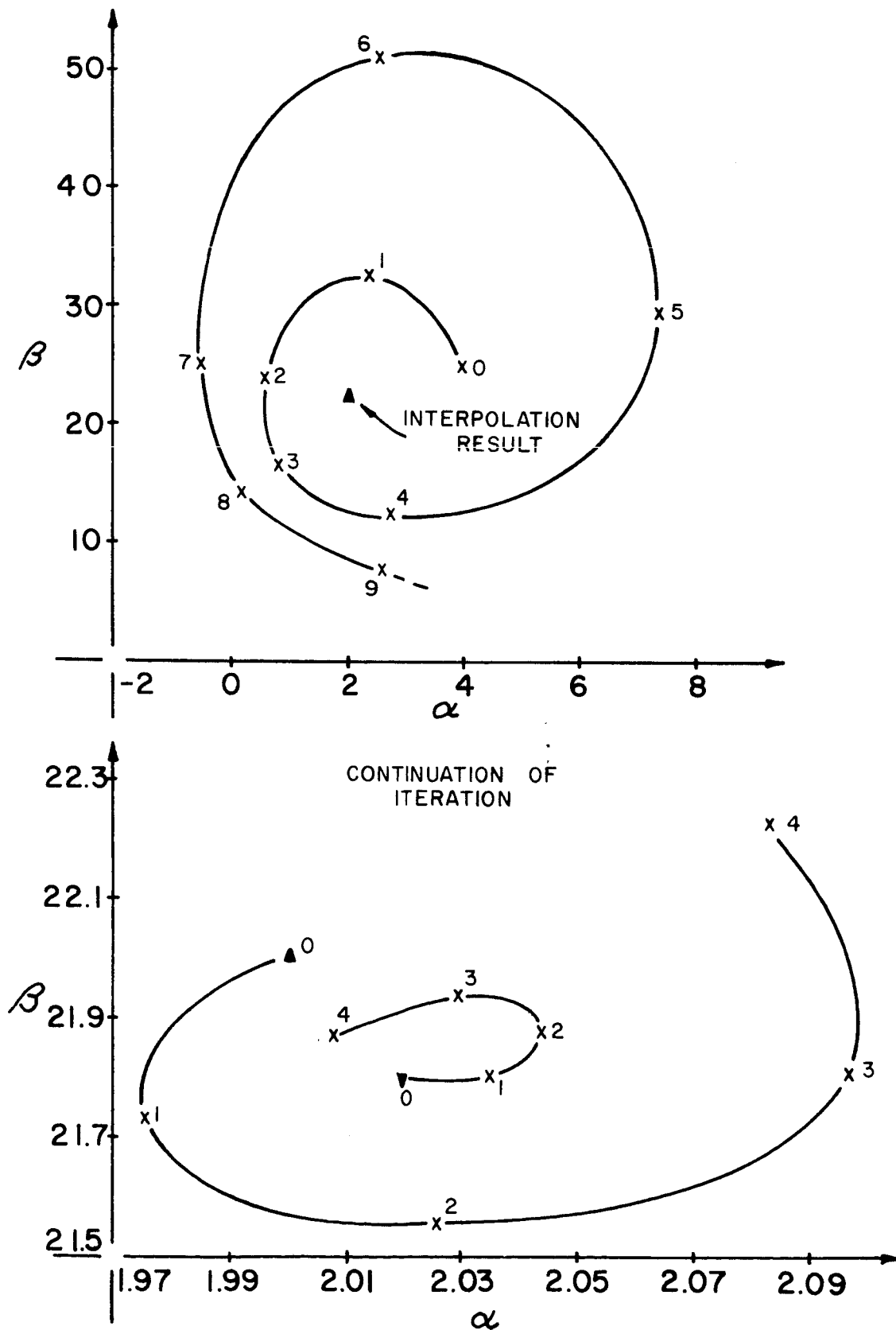


Figure 21 Points of Successive Iterations for the Determination of the Roots of the Quartic $z^4 - 2z^3 + 27z^2 - 61z + 291 = 0$

opposite λ 's, that the negative is greater than one in absolute value and the positive smaller than one.

It is however necessary to have as much as nine or ten iterations in order to get an idea of the asymptotic direction of the curve on which the points of the iteration lie. (It is to be remembered that the curve does not cross its asymptote which passes through the searched value.)

Applying this same procedure a few more times the coefficients of the searched quadratic factor can be determined with any desired accuracy. It is to be remembered that once the asymptotic direction has been determined, as has been in the case of the quartic of Equation (136) it is not necessary to use as many iterations in order to increase the accuracy.

The procedure has been illustrated in Table 11 and Figure 22 and the quadratic factor obtained with six significant digits which yields for the quartic of Equation (136)

$$(z^2 - 0.801989z - 74.1877)(z^2 + 7.80199z + 16.4447) = 0 \quad (137)$$

Since in engineering problems such an accuracy is usually not required, it would have been sufficient to make only two sets of iterations.

More examples of diverging cases could be considered, but it is believed that they would not bring anymore information about the method.

Table 11

Results of Successive Iterations for the Quartic $z^4 + 7z^3 - 64z^2 - 592z - 1220 = 0$

First Set of Iterations									
Iteration	0	1	2	3	4	5	6	7	8
α	7	-4.3	3.6	-4.8	2.7	-5.4	2.5	-6.0	2.6
β	-64	-162	-57	-128	-53	-121	-50	-121	-48
Second Set of Iterations									
Iteration	0	1	2	3	4	5			
α	-1	-0.68	-1.00	-0.64	-1.01	-0.60			
β	-75	-72.6	-75.4	-72.6	-75.7	-72.4			
Third Set of Iterations									
Iteration	0	1	2	3	4	5			
α	-0.800	-0.806	-0.799	-0.806	-0.798	-0.807			
β	-74.150	-74.208	-74.152	-74.216	-74.151	-74.222			
Fourth Set of Iterations									
Iteration	0	1	2	3					
α	-0.8020	-0.8023	-0.8019	-0.8022					
β	-74.185	-74.188	-74.186	-74.187					

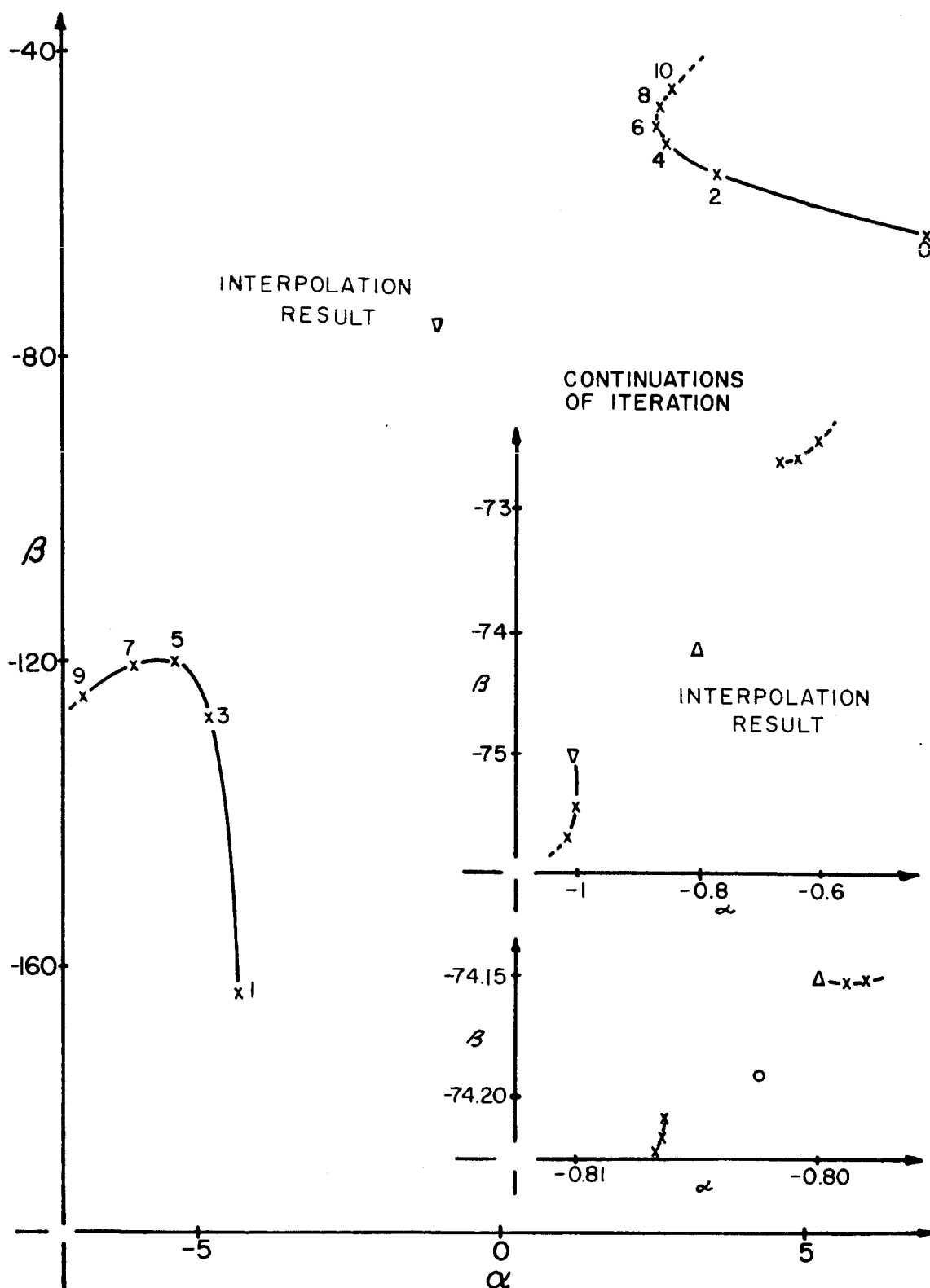


Figure 22

Points of Successive Iterations for the Determination of the Roots of the Quartic $z^4 + 7z^3 - 64z^2 - 592z - 1220 = 0$

SUMMARY, CONCLUSIONS AND RECOMMENDATIONS

Convergence of Oldenburger's Right to Left Synthetic Division for finding quadratic factors of polynomials is investigated. General forms for the successive terms of the iteration are obtained, and these are related by matrix equations. Convergence conditions are determined, and a graphical interpretation given in the case of cubics and quartics.

Comparison is made with the conditions of convergence of the same method when linear factors of polynomials are sought, and the influence of the stability of the polynomial in the sense of Routh-Hurwitz is considered.

The asymptotic convergence of Oldenburger's process for quadratic factors is studied and the possibility of increasing the speed of the process investigated.

The possibility of extending the use of the method to iterations which initially diverge is also considered.

Approaching the problem with a linear approximation of the iteration and using theorems on infinite powers of matrices proven by Oldenburger in an earlier work the following results were found:

1. The existence of convergence is shown to depend upon the magnitude of the characteristic roots of the matrix

relating successive iterations.

2. The stability in the sense of Routh-Hurwitz of the given polynomial is shown not to intervene in the convergence conditions.

3. Similarity between the convergence conditions obtained when the method is used for finding linear or quadratic factors is shown.

4. The different types of convergence are analyzed in Chapter 2 as a function of the Characteristic Roots of the matrix and the nature of the roots of the sought quadratic factor. Graphical interpretations are also given.

5. Ways of accelerating the convergence of the method are given and illustrated with examples in Chapter 3.

6. Finally, the cases for which the method does not converge are discussed in Chapter 4 where it was shown how in some cases by successive approximations the method can still be used to find quadratic factors in spite of its diverging nature.

The method is thus quite powerful and advantageous to use in hand and slide rule calculations. It can be very easily programmed on a digital computer with the amount of instructions being relatively low, and a digital computer has been used by the author in part of this work.

The main weakness of this work is the assumption of linearization which was made in order to study the asymptotical behaviour of the convergence of the process. Although it was

shown that in the general case this assumption remains valid even when the error vector is quite big, it would be interesting to consider what happens in the limit of validity of the hypothesis of linearity.

In the present work the matrix relating two successive points of the iteration is always defined as a function of the final values. It might be worth trying to express it as a function of the values obtained at two or three successive points of the iteration.

Another interesting extension is suggested by the similarities between the convergence conditions obtained in the case of the search of linear and quadratic factors. It might be interesting to study from a theoretical point of view the convergence to factors of order higher than two (cubics or quartics for instance). Examples which considered the case of the search for a cubic factor seemed to show that the convergence condition would be given by a relation having a form similar to those found for linear and quadratic factors although no theoretical study was made. The possibility of searching for cubics could be advantageous for studying higher order polynomials in which three roots are approximately equal. It could also be used to break a sextic into two cubics. However this method would be limited to equations having at least one real root which can be determined in a different manner; furthermore a graphical interpretation would not be helpful anymore for speeding up the results and

such an extension may prove to be only of an academic value without any direct application to everyday engineering use.

It might be also worth spending more time on the digital computer applications of the method. When the method was programmed in the course of this work, no provision was made to let the computer recognize by itself a slowly converging or a diverging case and automatically make the necessary logical decisions in order to speed up the solution. A particular study of this aspect of the problem could yield very profitable results.

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BIBLIOGRAPHY

APPENDIX A

Determination of b_k , b_k^* , α_1 , β_1 as

Functions of α , β and the Polynomial Coefficients

APPENDIX A

Determination of b_k , b_k^* , α_1 , β_1 as

Functions of α , β and the Polynomial Coefficients

In Equation (7) it was found that the following relation held between b_k , b_{k-1} and b_{k-2}

$$a_k = b_k + \frac{\alpha}{\beta} b_{k-1} + \frac{1}{\beta} b_{k-2}$$

$$k = 0, 1, \dots, n \quad (138)$$

b_{-1} and b_{-2} both zero

This relation can be used to express b_k as a function of b_{k-1} , b_{k-2} , i.e.

$$b_k = a_k - \frac{\alpha}{\beta} b_{k-1} - \frac{1}{\beta} b_{k-2} \quad (139)$$

which yields for the individual terms

$$\begin{aligned} b_0 &= a_0 \\ b_1 &= a_1 - \frac{\alpha}{\beta} b_0 \\ b_2 &= a_2 - \frac{\alpha}{\beta} b_1 - \frac{b_0}{\beta} \\ b_3 &= a_3 - \frac{\alpha}{\beta} b_2 - \frac{b_1}{\beta} \\ &\vdots \\ b_k &= a_k - \frac{\alpha}{\beta} b_{k-1} - \frac{b_{k-2}}{\beta} \\ b_n &= a_n - \frac{\alpha}{\beta} b_{n-1} - \frac{b_{n-2}}{\beta} \end{aligned} \quad (140)$$

Replacing successively in each equation the b_k terms by their expressions in a_k , there results

$$b_0 = a_0 \quad (141)$$

$$b_1 = a_1 - \frac{\alpha}{\beta} b_0 \quad (142)$$

$$b_1 = - \begin{vmatrix} a_0 & a_1 \\ 1 & \frac{\alpha}{\beta} \end{vmatrix} \quad (143)$$

$$b_2 = a_2 - \frac{\alpha}{\beta} a_1 + \left(\frac{\alpha^2}{\beta^2} - 1 \right) a_0 \quad (144)$$

$$b_2 = \begin{vmatrix} a_0 & a_1 & a_2 \\ 1 & \frac{\alpha}{\beta} & \frac{1}{\beta} \\ 0 & 1 & \frac{\alpha}{\beta} \end{vmatrix} \quad (145)$$

$$b_3 = a_3 - \frac{\alpha}{\beta} \begin{vmatrix} a_0 & a_1 & a_2 \\ 1 & \frac{\alpha}{\beta} & \frac{1}{\beta} \\ 0 & 1 & \frac{\alpha}{\beta} \end{vmatrix} + \frac{1}{\beta} \begin{vmatrix} a_0 & a_1 \\ 1 & \frac{\alpha}{\beta} \end{vmatrix} \quad (146)$$

$$b_3 = - \begin{vmatrix} a_0 & a_1 & a_2 & a_3 \\ 1 & \frac{\alpha}{\beta} & \frac{1}{\beta} & . \\ . & 1 & \frac{\alpha}{\beta} & \frac{1}{\beta} \\ . & . & 1 & \frac{\alpha}{\beta} \end{vmatrix} \quad (147)$$

where the missing terms represent zeros.

Similarly it is found that

$$b_4 = \begin{vmatrix} a_0 & a_1 & a_2 & a_3 & a_4 \\ 1 & \frac{\alpha}{\beta} & \frac{1}{\beta} & . & . \\ . & 1 & \frac{\alpha}{\beta} & \frac{1}{\beta} & . \\ . & . & 1 & \frac{\alpha}{\beta} & \frac{1}{\beta} \\ . & . & . & 1 & \frac{\alpha}{\beta} \end{vmatrix} \quad (148)$$

So far the b_k 's have been formed of determinants of $(k + 1)$ rows and columns, where

- The first row contains the a_0 to a_k terms
- The main diagonal $\frac{\alpha}{\beta}$ terms
- The diagonal on the right of it of $\frac{1}{\beta}$ terms
- The one on the left of it of 1's
- The sign in front of each determinant corresponds to $(-1)^k$

Assume that this form checks up to the $k-2$ and $k-1$ order, that is,

$$b_{k-2} = (-1)^{k-2} \begin{vmatrix} a_0 & a_1 & a_2 & \dots & a_{k-2} \\ 1 & \frac{\alpha}{\beta} & \frac{1}{\beta} & . & . \\ . & 1 & \frac{\alpha}{\beta} & \frac{1}{\beta} & . \\ . & . & \dots & \dots & \dots \\ . & . & . & 1 & \frac{\alpha}{\beta} \end{vmatrix} \quad (149)$$

and

$$b_{k-1} = (-1)^{k-1} \begin{vmatrix} a_0 & a_1 & a_2 & \dots & a_{k-2} & a_{k-1} \\ 1 & \frac{\alpha}{\beta} & \frac{1}{\beta} & . & . & . \\ . & 1 & \frac{\alpha}{\beta} & \frac{1}{\beta} & . & . \\ . & . & \dots & \dots & . & . \\ . & . & . & 1 & \frac{\alpha}{\beta} & \frac{1}{\beta} \\ . & . & . & . & 1 & \frac{\alpha}{\beta} \end{vmatrix} \quad (150)$$

Define as μ_k the determinant

$$\mu_k = (-1)^k \begin{vmatrix} a_0 & a_1 & a_2 & \dots & a_{k-1} & a_k \\ 1 & \frac{\alpha}{\beta} & \frac{1}{\beta} & . & . & . \\ . & 1 & \frac{\alpha}{\beta} & \frac{1}{\beta} & . & . \\ . & . & \dots & \dots & . & . \\ . & . & . & 1 & \frac{\alpha}{\beta} & \frac{1}{\beta} \\ . & . & . & . & 1 & \frac{\alpha}{\beta} \end{vmatrix} \quad (151)$$

Expanding μ_k along its last row

$$\mu_k = (-1)^k \left\{ \frac{\alpha}{\beta} \begin{vmatrix} a_0 & a_1 & \dots & \dots & a_{k-1} \\ 1 & \frac{\alpha}{\beta} & \frac{1}{\beta} & . & . \\ . & \dots & \dots & . & . \\ . & . & 1 & \frac{\alpha}{\beta} & \frac{1}{\beta} \\ . & . & . & 1 & \frac{\alpha}{\beta} \end{vmatrix} \right. \quad (152)$$

$$- \begin{vmatrix} a_0 & a_1 & \dots & a_{k-2} & a_k \\ 1 & \frac{\alpha}{\beta} & \frac{1}{\beta} & . & . \\ . & \dots & \dots & . & . \\ . & . & 1 & \frac{\alpha}{\beta} & 0 \\ . & . & . & 1 & \frac{1}{\beta} \end{vmatrix} \quad \left. \right\}$$

Expanding the second determinant of Equation (152)

$$\mu_k = (-1)^k \left\{ \frac{\alpha}{\beta} \begin{vmatrix} a_0 & a_1 & \dots & \dots & a_{k-1} \\ 1 & \frac{\alpha}{\beta} & \frac{1}{\beta} & \cdot & \cdot \\ \cdot & \dots & \dots & \cdot & \cdot \\ \cdot & \cdot & 1 & \frac{\alpha}{\beta} & \frac{1}{\beta} \\ \cdot & \cdot & \cdot & 1 & \frac{\alpha}{\beta} \end{vmatrix} \right. \quad (153)$$

$$- \frac{1}{\beta} \begin{vmatrix} a_0 & a_1 & \dots & \dots & a_{k-2} \\ 1 & \frac{\alpha}{\beta} & \frac{1}{\beta} & \cdot & \cdot \\ \cdot & \dots & \dots & \cdot & \cdot \\ \cdot & \cdot & 1 & \frac{\alpha}{\beta} & \frac{1}{\beta} \\ \cdot & \cdot & \cdot & 1 & \frac{\alpha}{\beta} \end{vmatrix} + \begin{vmatrix} a_0 & a_1 & \dots & a_{k-3} & a_k \\ 1 & \frac{\alpha}{\beta} & \frac{1}{\beta} & \cdot & \cdot \\ \cdot & \dots & \dots & \cdot & \cdot \\ \cdot & \cdot & 1 & \frac{\alpha}{\beta} & 0 \\ \cdot & \cdot & \cdot & 1 & 0 \end{vmatrix} \left. \right\}$$

$$\mu_k = (-1)^k \left\{ (-1)^{k-1} \frac{\alpha}{\beta} b_{k-1} - (-1)^{k-2} \frac{1}{\beta} b_{k-2} \right. \quad (154)$$

$$\left. (-1)^{k-2} a_k \right\}$$

$$\mu_k = - \frac{\alpha}{\beta} b_{k-1} - \frac{1}{\beta} b_{k-2} + a_k \quad (155)$$

$$\mu_k = a_k - \frac{\alpha}{\beta} b_{k-1} - \frac{1}{\beta} b_{k-2} \quad (156)$$

It can be seen that μ_k satisfies Equation (7), therefore

$$\mu_k \equiv b_k \quad (157)$$

and the general form assumed for the b_k 's will be valid.

Coefficients of the New Trial Factor

It has been found in Equations (8) and (9) that

$$\begin{aligned}\rho_1 &= b_{n-2} \\ \alpha_1 &= a_{n-1} - \frac{b_{n-3}}{\rho}\end{aligned}\tag{158}$$

ρ_1 is obtained at once from Equation (151) in which $k = n-2$ is considered.

This yields therefore:

$$\rho_1 = (-1)^{n-2} \begin{vmatrix} a_0 & a_1 & \dots & a_{n-3} & a_{n-2} \\ 1 & \frac{\alpha}{\rho} & \frac{1}{\rho} & \cdot & \cdot \\ \cdot & \dots & \dots & \cdot & \cdot \\ \cdot & \cdot & 1 & \frac{\alpha}{\rho} & \frac{1}{\rho} \\ \cdot & \cdot & \cdot & 1 & \frac{\alpha}{\rho} \end{vmatrix}\tag{159}$$

Considering the case $k = n-3$, there follows

$$b_{n-3} = (-1)^{n-3} \begin{vmatrix} a_0 & a_1 & \dots & a_{n-4} & a_{n-3} \\ 1 & \frac{\alpha}{\rho} & \frac{1}{\rho} & \cdot & \cdot \\ \cdot & \dots & \dots & \cdot & \cdot \\ \cdot & \cdot & 1 & \frac{\alpha}{\rho} & \frac{1}{\rho} \\ \cdot & \cdot & \cdot & 1 & \frac{\alpha}{\rho} \end{vmatrix}\tag{160}$$

Dividing by ρ and changing sign Equation (160) becomes

$$-\frac{b_{n-3}}{\beta} = (-1)^{n-2} \begin{vmatrix} a_0 & a_1 & \dots & a_{n-3} & 0 \\ 1 & \frac{\alpha}{\beta} & \frac{1}{\beta} & . & . \\ . & \dots & \dots & . & . \\ . & . & 1 & \frac{\alpha}{\beta} & 0 \\ . & . & . & 1 & \frac{1}{\beta} \end{vmatrix} \quad (161)$$

Therefore, there follows

$$\alpha_1 = a_{n-1} - \frac{b_{n-3}}{\beta} \quad (162)$$

$$\alpha_1 = (-1)^{n-2} \begin{vmatrix} a_0 & a_1 & \dots & a_{n-3} & a_{n-1} \\ 1 & \frac{\alpha}{\beta} & \frac{1}{\beta} & . & . \\ . & \dots & \dots & . & . \\ . & . & 1 & \frac{\alpha}{\beta} & 0 \\ . & . & . & 1 & \frac{1}{\beta} \end{vmatrix} \quad (163)$$

Terms of the Exact Quotient b_k^*

These terms are obtained by replacing in the expressions for b_k the quantities $\frac{\alpha}{\beta}$ and $\frac{1}{\beta}$ by $\frac{A}{B}$ and $\frac{1}{B}$ respectively, this yields

$$b_k^* = (-1)^k \begin{vmatrix} a_0 & a_1 & a_2 & \dots & a_{k-1} & a_k \\ 1 & \frac{A}{B} & \frac{1}{B} & . & . & . \\ . & 1 & \frac{A}{B} & \frac{1}{B} & . & . \\ . & . & \dots & \dots & . & . \\ . & . & . & 1 & \frac{A}{B} & \frac{1}{B} \\ . & . & . & . & 1 & \frac{A}{B} \end{vmatrix} \quad (164)$$

APPENDIX B

Determination of $h_k, x_{1, u+1}, x_{2, u+1}$

as Functions of $x_{1, u}, x_{2, u}$

APPENDIX B

Determination of $h_k, x_1, u+1, x_2, u+1$ as Functions of x_1, u, x_2, u

Consider Equation (139)

$$b_k = a_k - \frac{\alpha}{\beta} b_{k-1} - \frac{1}{\beta} b_{k-2}$$

Substituting for the b_k 's, α and β the values found in Equations (19) and (20)

$$b_k^* - h_k = a_k - \frac{A}{B} \left(1 - \frac{x_1}{A}\right) \left(1 + \frac{x_2}{B}\right) (b_{k-1}^* - h_{k-1}) - \frac{1}{B} \left(1 + \frac{x_2}{B}\right) (b_{k-2}^* - h_{k-2}) \quad (165)$$

or

$$b_k^* - h_k = a_k - \frac{A}{B} b_{k-1}^* - \frac{1}{B} b_{k-2}^* + \frac{A}{B} h_{k-1} + \frac{1}{B} h_{k-2} + \frac{b_{k-1}^*}{B} x_1 - \frac{A}{B} \frac{b_{k-1}^*}{B} x_2 - \frac{1}{B} \frac{b_{k-1}^*}{B} x_2 + f \left\{ \left(\frac{x_1}{A}\right)^2, \left(\frac{x_2}{A}\right)^2, \left(\frac{x_1}{A} \cdot \frac{x_2}{B}\right) \right\} \quad (166)$$

But according to Equation (15)

$$a_k = b_k^* + \frac{A}{B} b_{k-1}^* + \frac{1}{B} b_{k-2}^*$$

and neglecting the terms of order higher than one in $\frac{x_1}{A}$ and $\frac{x_2}{B}$, there follows

$$h_k = -\frac{A}{B} h_{k-1} - \frac{1}{B} h_{k-2} - \frac{b_{k-1}^*}{B} x_1 + \frac{A}{B} \frac{b_{k-1}^*}{B} x_2 + \frac{1}{B} \frac{b_{k-2}^*}{B} x_2 \quad (167)$$

Considering the first terms

$$h_0 = 0 \quad (168)$$

$$h_1 = -\frac{b_0^*}{B} x_1 + \frac{A}{B} \frac{b_0^*}{B} x_2 \quad (169)$$

or

$$h_1 = -\frac{b_0^*}{B} x_1 + \left| \begin{array}{c|c} \frac{b_0^*}{B} & 0 \\ \hline 1 & \frac{A}{B} \end{array} \right| x_2 \quad (170)$$

$$h_2 = -\frac{A}{B} h_1 - \frac{b_1^*}{B} x_1 + \frac{A}{B} \frac{b_1^*}{B} x_2 + \frac{1}{B} \frac{b_0^*}{B} x_2 \quad (171)$$

or

$$h_2 = \frac{A}{B} \frac{b_0^*}{B} x_1 - \frac{A}{B} \frac{1}{1} \left| \begin{array}{c|c} \frac{b_0^*}{B} & 0 \\ \hline 1 & \frac{A}{B} \end{array} \right| x_2 - \frac{b_1^*}{B} x_1 + \frac{A}{B} \frac{b_1^*}{B} x_2 + \frac{1}{B} \frac{b_0^*}{B} x_2 \quad (172)$$

that is

$$h_2 = \left| \begin{array}{c|c} \frac{b_0^*}{B} & \frac{b_1^*}{B} \\ \hline 1 & \frac{A}{B} \end{array} \right| x_1 - \left| \begin{array}{c|c} \frac{b_0^*}{B} & \frac{b_1^*}{B} \\ \hline 1 & \frac{A}{B} \end{array} \right| x_2 \quad (173)$$

Similarly it would be found that

$$h_3 = (-1)^3 \left\{ \left| \begin{array}{c|c} \frac{b_0^*}{B} & \frac{b_1^*}{B} \\ \hline 1 & \frac{A}{B} \end{array} \right| x_1 - \left| \begin{array}{c|c} \frac{b_0^*}{B} & \frac{b_1^*}{B} \\ \hline 1 & \frac{A}{B} \end{array} \right| x_2 \right\} \quad (174)$$

and

$$h_4 = (-1)^4 \left\{ \begin{array}{c|ccc} \frac{b_0^*}{B} & \frac{b_1^*}{B} & \frac{b_2^*}{B} & \frac{b_3^*}{B} \\ \hline 1 & \frac{A}{B} & 1 & \cdot \\ \cdot & \frac{A}{B} & \frac{A}{B} & \frac{1}{B} \\ \cdot & \cdot & 1 & \frac{A}{B} \\ \cdot & \cdot & \cdot & 1 \end{array} \right\} \begin{array}{c} x_1 \\ - \\ x_2 \end{array} \begin{array}{c} 0 \\ \cdot \\ \cdot \\ \frac{1}{B} \\ \frac{A}{B} \end{array} \quad (175)$$

Once again similarities can be noticed in the $h_1, \dots, h_4$ terms and the assumption will be made that a general form can be found for the terms up to h_{k-2} and h_{k-1} , that is

$$h_{k-2} = (-1)^{k-2} \left\{ \begin{array}{c|ccc} \frac{b_0^*}{B} & \frac{b_1^*}{B} & \dots & \frac{b_{k-3}^*}{B} \\ \hline \dots & \cdot & \cdot & \cdot \\ \cdot & 1 & \frac{A}{B} & \frac{1}{B} \\ \cdot & \cdot & \cdot & \frac{A}{B} \end{array} \right\} \begin{array}{c} x_1 \\ - \\ x_2 \end{array} \begin{array}{c} 0 \\ \cdot \\ \cdot \\ \frac{1}{B} \\ \frac{A}{B} \end{array} \quad (176)$$

and

$$h_{k-1} = (-1)^{k-1} \left\{ \begin{array}{c|ccc} \frac{b_0^*}{B} & \frac{b_1^*}{B} & \dots & \frac{b_{k-2}^*}{B} \\ \dots & \dots & \dots & \dots \\ \cdot & 1 & \frac{A}{B} & \frac{1}{B} \\ \cdot & \cdot & 1 & \frac{A}{B} \end{array} \right\} \begin{array}{c|ccc} \frac{b_0^*}{B} & \frac{b_1^*}{B} & \dots & \frac{b_{k-2}^*}{B} \\ \dots & \dots & \dots & \dots \\ \cdot & \frac{A}{B} & \frac{1}{B} & \cdot \\ \cdot & \cdot & \cdot & 1 \end{array} \begin{array}{c|ccc} 0 & \cdot & \cdot & \frac{1}{B} \\ \cdot & \cdot & \cdot & \frac{A}{B} \end{array} \right\} x_2 \quad (177)$$

To see if a general term of the same form is valid for the h_k term, define ν_k

$$\nu_k = (-1)^k \left\{ \begin{array}{c|ccc} \frac{b_0^*}{B} & \frac{b_1^*}{B} & \dots & \frac{b_{k-1}^*}{B} \\ \dots & \dots & \dots & \dots \\ \cdot & 1 & \frac{A}{B} & \frac{1}{B} \\ \cdot & \cdot & 1 & \frac{A}{B} \end{array} \right\} \begin{array}{c|ccc} \frac{b_0^*}{B} & \frac{b_1^*}{B} & \dots & \frac{b_{k-1}^*}{B} \\ \dots & \dots & \dots & \dots \\ \cdot & \frac{A}{B} & \frac{1}{B} & \cdot \\ \cdot & \cdot & \cdot & 1 \end{array} \begin{array}{c|ccc} 0 & \cdot & \cdot & \frac{1}{B} \\ \cdot & \cdot & \cdot & \frac{A}{B} \end{array} \right\} x_2 \quad (178)$$

Expanding the determinants of v_k along their last rows

$$v_k = (-1)^{k-1} \left\{ (-1)^k \frac{b_{k-1}}{B} x_1 + \frac{A}{B} \begin{vmatrix} \frac{b_0^*}{B} & \frac{b_1^*}{B} & \dots & \frac{b_{k-2}^*}{B} \\ 1 & \frac{A}{B} & \dots & \frac{1}{B} \\ \dots & \dots & \dots & \dots \\ \frac{b_0^*}{B} & \frac{b_1^*}{B} & \dots & \frac{b_{k-3}^*}{B} \end{vmatrix} x_1 - \begin{vmatrix} \frac{b_0^*}{B} & \frac{b_1^*}{B} & \dots & \frac{b_{k-2}^*}{B} \\ 1 & \frac{A}{B} & \dots & \frac{1}{B} \\ \dots & \dots & \dots & \dots \\ \frac{b_0^*}{B} & \frac{b_1^*}{B} & \dots & \frac{b_{k-3}^*}{B} \end{vmatrix} \frac{A}{B} \right\}$$

$$- \frac{A}{B} \left\{ \begin{vmatrix} \frac{b_0^*}{B} & \frac{b_1^*}{B} & \dots & \frac{b_{k-2}^*}{B} \\ 1 & \frac{A}{B} & \dots & \frac{1}{B} \\ \dots & \dots & \dots & \dots \\ \frac{b_0^*}{B} & \frac{b_1^*}{B} & \dots & \frac{b_{k-2}^*}{B} \end{vmatrix} x_2 + \frac{1}{B} \begin{vmatrix} \frac{b_0^*}{B} & \frac{b_1^*}{B} & \dots & \frac{b_{k-2}^*}{B} \\ 1 & \frac{A}{B} & \dots & \frac{1}{B} \\ \dots & \dots & \dots & \dots \\ \frac{b_0^*}{B} & \frac{b_1^*}{B} & \dots & \frac{b_{k-2}^*}{B} \end{vmatrix} x_2 \right\} \quad (179)$$

$$\begin{aligned}
v_k &= (-1)^k \left\{ \frac{A}{B} \left[\begin{array}{c|ccc} \frac{b_0^*}{B} & \frac{b_1^*}{B} & \dots & \frac{b_{k-2}^*}{B} \\ \dots & \dots & \dots & \dots \\ \cdot & 1 & \frac{A}{B} & \frac{1}{B} \\ \cdot & \cdot & 1 & \frac{A}{B} \end{array} \right] x_1 - \left[\begin{array}{c|ccc} \frac{b_0^*}{B} & \frac{b_1^*}{B} & \dots & \frac{b_{k-1}^*}{B} \\ 1 & \frac{A}{B} & \frac{1}{B} & \cdot \\ \cdot & \dots & \dots & \cdot \\ \cdot & \cdot & 1 & \frac{1}{B} \\ \cdot & \cdot & \cdot & \frac{A}{B} \end{array} \right] x_2 \right\} \\
&- \frac{1}{B} \left[\begin{array}{c|ccc} \frac{b_0^*}{B} & \frac{b_1^*}{B} & \dots & \frac{b_{k-3}^*}{B} \\ \dots & \dots & \dots & \dots \\ \cdot & 1 & \frac{A}{B} & \frac{1}{B} \\ \cdot & \cdot & 1 & \frac{A}{B} \end{array} \right] x_1 - \left[\begin{array}{c|ccc} \frac{b_0^*}{B} & \frac{b_1^*}{B} & \dots & \frac{b_{k-2}^*}{B} \\ 1 & \frac{A}{B} & \frac{1}{B} & \cdot \\ \cdot & \dots & \dots & \cdot \\ \cdot & \cdot & 1 & \frac{1}{B} \\ \cdot & \cdot & \cdot & \frac{A}{B} \end{array} \right] x_2 + (-1)^{k-1} \frac{b_{k-1}^*}{B} x_1 \right\} \quad (180)
\end{aligned}$$

or

$$\nu_k = (-1)^k \left\{ \frac{A}{B} \left[\begin{array}{c|ccc} \frac{b_0^*}{B} & \frac{b_1^*}{B} & \dots & \frac{b_{k-2}^*}{B} \\ \hline \dots & \dots & \dots & \dots \\ \cdot & 1 & \frac{A}{B} & \frac{1}{B} \\ \cdot & \cdot & 1 & \frac{A}{B} \end{array} \right] x_1 - \left[\begin{array}{c|ccc} \frac{b_0^*}{B} & \frac{b_1^*}{B} & \dots & \frac{b_{k-2}^*}{B} \\ \hline 1 & \frac{A}{B} & \frac{1}{B} & \cdot \\ \cdot & \dots & \dots & \cdot \\ \cdot & \cdot & 1 & \frac{A}{B} \end{array} \right] x_2 - (-1)^{k-1} \frac{b_{k-1}^*}{B} x_2 \right]$$

(181)

$$- \frac{1}{B} \left[\begin{array}{c|ccc} \frac{b_0^*}{B} & \frac{b_1^*}{B} & \dots & \frac{b_{k-3}^*}{B} \\ \hline \dots & \dots & \dots & \dots \\ \cdot & 1 & \frac{A}{B} & \frac{1}{B} \\ \cdot & \cdot & 1 & \frac{A}{B} \end{array} \right] x_1 - \left[\begin{array}{c|ccc} \frac{b_0^*}{B} & \frac{b_1^*}{B} & \dots & \frac{b_{k-3}^*}{B} \\ \hline 1 & \frac{A}{B} & \frac{1}{B} & \cdot \\ \cdot & \dots & \dots & \cdot \\ \cdot & \cdot & 1 & \frac{A}{B} \end{array} \right] x_2 - (-1)^{k-2} \frac{b_{k-2}^*}{B} x_2 \right]$$

$$+ (-1)^{k-1} \frac{b_{k-1}^*}{B} x_1 \left. \vphantom{\frac{b_{k-1}^*}{B}} \right\}$$

or

$$\begin{aligned} \gamma_k = (-1)^k \left\{ \frac{A}{B} \left[(-1)^{k-1} h_{k-1} - (-1)^{k-1} \frac{b_{k-1}^*}{B} x_2 \right] \right. \\ \left. - \frac{1}{B} \left[(-1)^{k-2} h_{k-2} - (-1)^{k-2} \frac{b_{k-2}^*}{B} x_2 \right] + (-1)^{k-1} \frac{b_{k-1}^*}{B} x_1 \right\} \end{aligned} \quad (182)$$

whereupon

$$\gamma_k = -\frac{A}{B} h_{k-1} - \frac{1}{B} h_{k-2} - \frac{b_{k-1}^*}{B} x_1 + \frac{A}{B} \frac{b_{k-1}^*}{B} x_2 + \frac{1}{B} \frac{b_{k-2}^*}{B} x_2 \quad (183)$$

Therefore γ_k satisfies Equation (167) and

$$h_k = \gamma_k \quad (184)$$

It was found in Equations (8) and (9) that

$$\alpha_{u+1} = a_{n-1} - \frac{b_{n-3}}{\beta_u} \quad (185)$$

$$\beta_{u+1} = b_{n-2}$$

Substituting the values found in Equations (19) and (21) this yields

$$A - x_{1, u+1} = a_{n-1} - \frac{1}{B} \left(1 + \frac{x_{2,u}}{B} \right) (b_{n-3}^* - h_{n-3}) \quad (186)$$

$$B - x_{2, u+1} = b_{n-2}^* - h_{n-2}$$

Substituting for A and B the values obtained from Equations (6), (9) and (16) and neglecting the terms of order higher than one in x_1 and x_2

$$x_{1, u+1} = -\frac{1}{B} h_{n-3} + \frac{1}{B} \frac{b_{n-3}^*}{B} x_{2,u} \quad (187)$$

$$x_{2, u+1} = h_{n-2}$$

which yields, by making $k = n-2$ in Equation (178)

$$\begin{aligned}
 x_{2, u+1} &= (-1)^{n-2} \left\{ \begin{array}{c|c} \begin{array}{cccc} \frac{b_0^*}{B} & \frac{b_1^*}{B} & \dots & \frac{b_{n-3}^*}{B} \\ \dots & \dots & \dots & \dots \\ \cdot & 1 & \frac{A}{B} & \frac{1}{B} \\ \cdot & \cdot & 1 & \frac{A}{B} \end{array} & \begin{array}{c|c} \begin{array}{cccc} \frac{b_0^*}{B} & \frac{b_1^*}{B} & \dots & \frac{b_{n-3}^*}{B} \\ 1 & \frac{A}{B} & \frac{1}{B} & \cdot \\ \cdot & \dots & \dots & \cdot \\ \cdot & \cdot & 1 & \frac{1}{B} \end{array} & \begin{array}{c} 0 \\ \cdot \\ \cdot \\ \frac{A}{B} \end{array} \end{array} \right\} x_{1,u} - \left\{ \begin{array}{c|c} \begin{array}{cccc} \frac{b_0^*}{B} & \frac{b_1^*}{B} & \dots & \frac{b_{n-3}^*}{B} \\ \dots & \dots & \dots & \dots \\ \cdot & 1 & \frac{A}{B} & \frac{1}{B} \\ \cdot & \cdot & 1 & \frac{A}{B} \end{array} & \begin{array}{c} 0 \\ \cdot \\ \cdot \\ \frac{A}{B} \end{array} \end{array} \right\} x_{2,u}
 \end{aligned}
 \tag{188}$$

and

$$\begin{aligned}
 -x_{1, u+1} &= (-1)^{n-3} \frac{1}{B} \left\{ \begin{array}{c|c} \begin{array}{cccc} \frac{b_0^*}{B} & \frac{b_1^*}{B} & \dots & \frac{b_{n-4}^*}{B} \\ \dots & \dots & \dots & \dots \\ \cdot & 1 & \frac{A}{B} & \frac{1}{B} \\ \cdot & \cdot & 1 & \frac{A}{B} \end{array} & \begin{array}{c|c} \begin{array}{cccc} \frac{b_0^*}{B} & \frac{b_1^*}{B} & \dots & \frac{b_{n-4}^*}{B} \\ 1 & \frac{A}{B} & \frac{1}{B} & \cdot \\ \cdot & \dots & \dots & \cdot \\ \cdot & \cdot & 1 & \frac{1}{B} \end{array} & \begin{array}{c} 0 \\ \cdot \\ \cdot \\ \frac{A}{B} \end{array} \end{array} \right\} x_{1,u} - \left\{ \begin{array}{c|c} \begin{array}{cccc} \frac{b_0^*}{B} & \frac{b_1^*}{B} & \dots & \frac{b_{n-4}^*}{B} \\ \dots & \dots & \dots & \dots \\ \cdot & 1 & \frac{A}{B} & \frac{1}{B} \\ \cdot & \cdot & 1 & \frac{A}{B} \end{array} & \begin{array}{c} 0 \\ \cdot \\ \cdot \\ \frac{A}{B} \end{array} \end{array} \right\} x_{2,u} \\
 &\quad - \frac{1}{B} \frac{b_{n-3}^*}{B} x_{2,u}
 \end{aligned}
 \tag{189}$$

(190)

$$\begin{aligned}
 -x_{1,u+1} &= (-1)^{n-3} \frac{1}{B} \left\{ \begin{array}{c|c} \begin{array}{cccc} \frac{b_0^*}{B} & \frac{b_1^*}{B} & \dots & \frac{b_{n-4}^*}{B} \\ \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots \end{array} & \begin{array}{c} x_{1,u} \\ \dots \\ \dots \\ \dots \end{array} \\ \hline \begin{array}{cccc} \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots \end{array} & \begin{array}{c} x_{2,u} \\ \dots \\ \dots \\ \dots \end{array} \end{array} \right.
 \end{aligned}$$

$$- (-1)^{n-3} \frac{1}{B} \frac{b_{n-3}^*}{B} x_{2,u}$$

(191)

$$\begin{aligned}
 x_{1,u+1} &= (-1)^{n-2} \frac{1}{B} \left\{ \begin{array}{c|c} \begin{array}{cccc} \frac{b_0^*}{B} & \frac{b_1^*}{B} & \dots & \frac{b_{n-4}^*}{B} \\ \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots \end{array} & \begin{array}{c} x_{1,u} \\ \dots \\ \dots \\ \dots \end{array} \\ \hline \begin{array}{cccc} \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots \end{array} & \begin{array}{c} x_{2,u} \\ \dots \\ \dots \\ \dots \end{array} \end{array} \right.
 \end{aligned}$$

Equations (191) and (188) can then be written in the form of Equation (25) or (30) which gave the matrix relation between the successive error vectors.

APPENDIX C

Determination of the Characteristic Roots of the M Matrix

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Determination of the Characteristic Roots of the M Matrix

It was shown in Equation (25) that the components of the error vectors in two successive iterations were related by:

$$x_{1, u+1} = m_{11} x_{1,u} + m_{12} x_{2,u}$$

$$x_{2, u+1} = m_{21} x_{1,u} + m_{22} x_{2,u}$$

It was also shown that these equations could be rewritten in the matrix form:

$$\begin{bmatrix} x_{1, u+1} \\ x_{2, u+1} \end{bmatrix} = \begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix} \begin{bmatrix} x_{1,u} \\ x_{2,u} \end{bmatrix}$$

In order for the iteration to converge and Equation (32) to be satisfied, it is necessary to solve Equation (33) which can also be rewritten as

$$\begin{bmatrix} m_{11} - \lambda & m_{12} \\ m_{21} & m_{22} - \lambda \end{bmatrix} = 0 \quad (192)$$

that is

$$(m_{11} - \lambda)(m_{22} - \lambda) - m_{12} m_{21} = 0 \quad (193)$$

or

$$\lambda^2 - (m_{11} + m_{22})\lambda + m_{11} m_{22} - m_{12} m_{21} = 0 \quad (194)$$

the solution of which is

$$\lambda = \frac{1}{2} \left\{ (m_{11} + m_{22}) \pm \sqrt{(m_{11} + m_{22})^2 - 4(m_{11} m_{22} - m_{12} m_{21})} \right\} \quad (195)$$

It is now necessary to evaluate Equation (195) replacing the m 's by their expressions found in Equations (26) to (29).

$$m_{11} + m_{22} = (-1)^{n-1} \left\{ \begin{array}{c} \left| \begin{array}{cccc} \frac{b_0^*}{B} & \frac{b_1^*}{B} & \dots & \frac{b_{n-3}^*}{B} & 0 \\ 1 & \frac{A}{B} & \frac{1}{B} & \cdot & \cdot \\ \cdot & \dots & \dots & \dots & \cdot \\ \cdot & \cdot & 1 & \frac{A}{B} & \frac{1}{B} \\ \cdot & \cdot & \cdot & 1 & \frac{A}{B} \end{array} \right| - \left| \begin{array}{cccc} \frac{b_0^*}{B} & \dots & \frac{b_{n-4}^*}{B} & 0 \\ \dots & \dots & \dots & \cdot \\ \cdot & 1 & \frac{A}{B} & 0 \\ \cdot & \cdot & \cdot & \frac{1}{B} \end{array} \right| \end{array} \right\} \quad (196)$$

$$m_{11} + m_{22} = (-1)^{n-1} \left\{ \frac{A}{B} \left| \begin{array}{cccc} \frac{b_0^*}{B} & \frac{b_1^*}{B} & \dots & \frac{b_{n-4}^*}{B} & 0 \\ 1 & \frac{A}{B} & \frac{1}{B} & \cdot & \cdot \\ \cdot & \dots & \dots & \dots & \cdot \\ \cdot & \cdot & 1 & \frac{A}{B} & \frac{1}{B} \\ \cdot & \cdot & \cdot & 1 & \frac{A}{B} \end{array} \right| - 2 \left| \begin{array}{cccc} \frac{b_0^*}{B} & \dots & \frac{b_{n-4}^*}{B} & 0 \\ \dots & \dots & \dots & \cdot \\ \cdot & 1 & \frac{A}{B} & 0 \\ \cdot & \cdot & \cdot & \frac{1}{B} \end{array} \right| \right\} \quad (197)$$

To determine the terms under the radical sign, first by squaring (197), one

Similarly

$$-4 m_{11} m_{22} = 4 \frac{A}{B}$$

$\frac{b_0^*}{B}$	$\frac{b_1^*}{B}$	$\dots$	$\dots$	$\frac{b_{n-3}^*}{B}$	$\frac{b_0^*}{B}$	$\frac{b_1^*}{B}$	$\dots$	$\dots$	$\frac{b_{n-4}^*}{B}$	0
1	$\frac{A}{B}$	$\frac{1}{B}$	$\dots$	$\dots$	1	$\frac{A}{B}$	$\frac{1}{B}$	$\dots$	$\frac{1}{B}$	$\dots$
$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$
$\dots$	$\dots$	1	$\frac{A}{B}$	$\frac{1}{B}$	$\dots$	$\dots$	1	$\frac{A}{B}$	0	0
$\dots$	$\dots$	$\dots$	1	$\frac{A}{B}$	$\dots$	$\dots$	$\dots$	0	$\frac{1}{B}$	$\frac{1}{B}$

(199)

$$\begin{array}{ccccccc}
b_0^* & b_1^* & \dots & b_{n-4}^* & 0 \\
\hline
\frac{b_0^*}{B} & \frac{b_1^*}{B} & \dots & \frac{b_{n-4}^*}{B} & 0 \\
1 & \frac{A}{B} & \frac{1}{B} & \cdot & \cdot \\
\cdot & \dots\dots\dots & \cdot & \cdot & \cdot \\
\cdot & \cdot & 1 & \frac{A}{B} & 0 \\
\cdot & \cdot & \cdot & 1 & \frac{1}{B}
\end{array}$$

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and

$$4 m_{12} m_{21} = -\frac{4}{B} \quad (200)$$

$\frac{b_0^*}{B}$	$\frac{b_1^*}{B}$	$\dots$	$\dots$	$\frac{b_{n-3}^*}{B}$
1	$\frac{A}{B}$	$\frac{1}{B}$	$\dots$	$\dots$
$\dots$	$\dots$	$\dots$	$\dots$	$\dots$
$\dots$	$\dots$	1	$\frac{A}{B}$	$\frac{1}{B}$
$\dots$	$\dots$	$\dots$	1	$\frac{A}{B}$

Adding up Equations (198), (199), (200), there results

$$(m_{11} + m_{22})^2 - 4 (m_{11} m_{22} - m_{12} m_{21}) = \left\{ \frac{A^2}{B^2} - \frac{4}{B} \right\} \quad (201)$$

$\frac{b_0^*}{B}$	$\frac{b_1^*}{B}$	$\dots$	$\dots$	$\frac{b_{n-3}^*}{B}$
1	$\frac{A}{B}$	$\frac{1}{B}$	$\dots$	$\dots$
$\dots$	$\dots$	$\dots$	$\dots$	$\dots$
$\dots$	$\dots$	1	$\frac{A}{B}$	$\frac{1}{B}$
$\dots$	$\dots$	$\dots$	1	$\frac{A}{B}$

which yields, when taking the square root:

$$\sqrt{(m_{11} + m_{22})^2 - 4(m_{11}m_{22} - m_{12}m_{21})} = \pm \frac{1}{B} \sqrt{A^2 - 4B} \quad (202)$$

$$\begin{array}{ccccccc} \frac{b_0^*}{B} & \frac{b_1^*}{B} & \dots & \frac{b_{n-4}^*}{B} & \frac{b_{n-3}^*}{B} & & \\ 1 & \frac{A}{B} & \frac{1}{B} & \cdot & \cdot & & \\ \cdot & \dots & \dots & \dots & \cdot & & \\ \cdot & \cdot & 1 & \frac{A}{B} & \frac{1}{B} & & \\ \cdot & \cdot & \cdot & 1 & \frac{A}{B} & & \end{array}$$

Combining (195), (197) and (202), there is obtained for λ

$$\lambda = (-1)^{n-1} \frac{1}{2} \left\{ \left[\frac{A}{B} \pm \frac{1}{B} \sqrt{A^2 - 4B} \right] \begin{array}{ccccccc} \frac{b_0^*}{B} & \frac{b_1^*}{B} & \dots & \frac{b_{n-3}^*}{B} & \frac{b_{n-4}^*}{B} & & \\ 1 & \frac{A}{B} & \frac{1}{B} & \cdot & \cdot & & \\ \cdot & \dots & \dots & \dots & \cdot & & \\ \cdot & \cdot & 1 & \frac{A}{B} & \frac{1}{B} & & \\ \cdot & \cdot & \cdot & 1 & \frac{A}{B} & & \end{array} - 2 \begin{array}{ccccccc} \frac{b_0^*}{B} & \frac{b_1^*}{B} & \dots & \frac{b_{n-4}^*}{B} & \frac{b_{n-3}^*}{B} & & \\ 1 & \frac{A}{B} & \frac{1}{B} & \cdot & \cdot & & \\ \cdot & \dots & \dots & \dots & \cdot & & \\ \cdot & \cdot & 1 & \frac{A}{B} & \frac{1}{B} & & \\ \cdot & \cdot & \cdot & 1 & \frac{A}{B} & & \end{array} \right\} \quad (203)$$

which can also be rewritten as:

$$\lambda = (-1)^{n-1} \frac{1}{B} \left\{ \left[A \pm \sqrt{A^2 - 4B} \right] \begin{array}{c|cccc} \frac{b_0^*}{B} & \frac{b_1^*}{B} & \dots & \dots & \frac{b_{n-3}^*}{B} \\ \hline 1 & \frac{A}{B} & \frac{1}{B} & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \end{array} \right\} \quad (204)$$

It can be noticed now that

$$z_1 = -\frac{1}{2} \left[A \pm \sqrt{A^2 - 4B} \right] \quad (205)$$

represents either root of the quadratic factor

$$z^2 + Az + B = (z - z_1)(z - z_2) \quad (206)$$

and therefore Equation (204) can be rewritten in the form

$$\lambda_i = (-1)^n \frac{1}{B} \left\{ z_i \left[\begin{array}{cccc} \frac{b_0^*}{B} & \frac{b_1^*}{B} & \dots & \dots & \frac{b_{n-3}^*}{B} \\ 1 & \frac{A}{B} & & & \cdot \\ \cdot & \dots & \dots & & \cdot \\ \cdot & \cdot & 1 & \frac{A}{B} & \frac{1}{B} \\ \cdot & \cdot & \cdot & 1 & \frac{A}{B} \end{array} \right] + \left[\begin{array}{cccc} \frac{b_0^*}{B} & \frac{b_1^*}{B} & \dots & \dots & \frac{b_{n-4}^*}{B} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & \frac{A}{B} & & \frac{1}{B} \\ \cdot & \cdot & 1 & & \frac{A}{B} \end{array} \right] \right\} \quad (207)$$

Using the results of Equations (28) and (26), it is possible to rewrite Equation (207)

$$\lambda_i = \frac{m_{21}}{B} z_i + m_{11} \quad (208)$$

or

$$\lambda_i = m_{11} - m_{12} z_i \quad (209)$$

If now the expression for λ_2 is considered by making $z_1 = z_2$ in Equation (207), one gets after expanding the determinants along their last rows

$$\lambda_2 = (-1)^n \frac{1}{B} \left\{ (z_2 \frac{A}{B} + 1) \begin{vmatrix} \frac{b_0^*}{B} & \frac{b_1^*}{B} & \dots & \frac{b_{n-4}^*}{B} \\ 1 & \frac{A}{B} & \frac{1}{B} & \cdot \\ \cdot & \dots & \dots & \cdot \\ \cdot & \frac{1}{B} & \frac{A}{B} & \frac{1}{B} \\ \cdot & \cdot & \cdot & \frac{A}{B} \end{vmatrix} - \frac{z_2}{B} \begin{vmatrix} \frac{b_0^*}{B} & \frac{b_1^*}{B} & \dots & \frac{b_{n-5}^*}{B} \\ \cdot & 1 & \frac{A}{B} & \frac{1}{B} \\ \cdot & \cdot & 1 & \frac{A}{B} \end{vmatrix} \right\} + (-1)^{n-3} \frac{b_{n-3}^*}{B}$$

But since

$$A = -(z_1 + z_2)$$

$$B = z_1 z_2$$

there follows

$$(z_2 \frac{A}{B} + 1) = - \frac{z_2}{z_1} \tag{212}$$

and

$$\frac{z_2}{B} = \frac{1}{z_1} \tag{213}$$

Therefore λ_2 becomes

$$\lambda_2 = (-1)^n \frac{1}{B} \left\{ \frac{z_2}{z_1} \begin{vmatrix} \frac{b_0^*}{B} & \frac{b_1^*}{B} & \dots & \dots & \frac{b_{n-4}^*}{B} \\ 1 & \frac{A}{B} & \frac{1}{B} & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & 1 & \frac{A}{B} & \frac{1}{B} \\ \dots & \dots & \dots & \dots & \frac{A}{B} \end{vmatrix} - \frac{1}{z_1} \begin{vmatrix} \frac{b_0^*}{B} & \frac{b_1^*}{B} & \dots & \dots & \frac{b_{n-5}^*}{B} \\ \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & 1 & \frac{A}{B} & \frac{1}{B} \\ \dots & \dots & \dots & \dots & \frac{A}{B} \end{vmatrix} \right\} - \frac{1}{z_1} \frac{b_{n-3}^*}{B} \tag{214}$$

or

$$\lambda_2 = \frac{1}{z_1} \left\{ (-1)^{n-1} \frac{1}{B} \left[z_2 \begin{vmatrix} \frac{b_0^*}{B} & \frac{b_1^*}{B} & \dots & \dots & \frac{b_{n-4}^*}{B} \\ 1 & \frac{A}{B} & \frac{1}{B} & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \end{vmatrix} + \begin{vmatrix} \frac{b_0^*}{B} & \frac{b_1^*}{B} & \dots & \dots & \frac{b_{n-5}^*}{B} \\ \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \end{vmatrix} \right] \right\} \quad (215)$$

$$- \frac{1}{z_1} \frac{b_{n-3}^*}{B}$$

It can be seen that there is an analogy between the term in brackets in the right hand side of Equation (215) and the expression for λ_2 obtained by making $z_1 = z_2$ in Equation (20); the only difference is that the orders of the determinants have been reduced by one unit, the sign of the expression has been changed and the whole bracket has been multiplied by $\frac{1}{z_1}$.

Therefore, in order to calculate this expression, one can apply the same procedure which has been used for the calculation of Equation (207).

This would yield:

$$\lambda_2 = \frac{1}{z_1^2} \left\{ (-1)^{n-2} \frac{1}{B} \left[z_2 \begin{vmatrix} \frac{b_0^*}{B} & \frac{b_1^*}{B} & \dots & \frac{b_{n-5}^*}{B} \\ 1 & \frac{A}{B} & \dots & \frac{1}{B} \\ \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots \end{vmatrix} + \begin{vmatrix} \frac{b_0^*}{B} & \frac{b_1^*}{B} & \dots & \frac{b_{n-6}^*}{B} \\ \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots \end{vmatrix} \right] \right\} \\ - \frac{1}{z_1} \frac{b_{n-3}^*}{B} - \frac{1}{z_1^2} \frac{b_{n-4}^*}{B} \quad (216)$$

Every time the same calculation is applied again to the expression of λ_2 the order of the determinants would be reduced by one, and it is found finally that

$$\lambda_2 = - \frac{1}{z_1} \frac{b_{n-3}^*}{B} - \frac{1}{z_1^2} \frac{b_{n-4}^*}{B} - \dots - \frac{1}{z_1} \frac{b_2^*}{B} \\ + \frac{1}{z_1^{n-4}} \left\{ (-1)^4 \frac{1}{B} \left[z_2 \begin{vmatrix} \frac{b_0^*}{B} & \frac{b_1^*}{B} & \dots & \frac{b_{n-6}^*}{B} \\ 1 & \frac{A}{B} & \dots & \frac{1}{B} \\ \dots & \dots & \dots & \dots \end{vmatrix} + \frac{b_0^*}{B} \right] \right\} \quad (217)$$

But

$$(-1)^4 \frac{1}{B} \left[z_2 \left| \frac{b_0^*}{B} \frac{1}{A} \frac{1}{B} \right| + \frac{b_0^*}{B} \right] = (-1)^4 \frac{1}{B} \left\{ \frac{b_0^*}{B} (z_2 \frac{A}{B} + 1) - \frac{b_1^*}{B} z_2 \right\} \quad (218)$$

or

$$(-1)^4 \frac{1}{B} \left[z_2 \left| \frac{b_0^*}{B} \frac{1}{A} \frac{1}{B} \right| + \frac{b_0^*}{B} \right] = \frac{1}{z_1} \frac{b_1^*}{B} - \frac{1}{z_1^2} \frac{b_0^*}{B} \quad (219)$$

Therefore λ_2 becomes

$$\lambda_2 = -\frac{1}{z_1} \frac{b_{n-3}^*}{B} - \frac{1}{z_1^2} \frac{b_{n-4}^*}{B} - \dots - \frac{1}{z_1^{n-4}} \frac{b_2^*}{B} - \frac{1}{z_1^{n-3}} \frac{b_1^*}{B} - \frac{1}{z_1^{n-2}} \frac{b_0^*}{B} \quad (220)$$

which can also be rewritten in the form

$$\lambda_2 = - \sum_{k=0}^{n-3} \frac{b_k^*}{B} \frac{1}{z_1^{n-2-k}} \quad (221)$$

or

$$\lambda_2 = - \frac{1}{z_1^{n-2}} \sum_{k=0}^{n-3} \frac{b_k^*}{B} z_1^k \quad (222)$$

Taking Equation (16) into consideration, λ_2 can be written

$$\lambda_2 = 1 - \frac{1}{z_1^{n-2}} \sum_{k=0}^{n-2} \frac{b_k^*}{B} z_1^k \quad (223)$$

But by comparison with Equations (13) and (14) it can be seen that

$$\sum_{k=0}^{n-2} \frac{b_k^*}{B} z^k = \frac{P_n(z)}{p^*(z)} \quad (224)$$

which will be denoted as

$$\sum_{k=0}^{n-2} \frac{b_k^*}{B} z^k = \prod_2(z) \quad (225)$$

and this yields

$$\lambda_2 = 1 - \frac{1}{z_1^{n-2}} \prod_2(z_1) \quad (226)$$

If $P_n(z)$ is expressed as

$$P_n(z) = (z - z_1)(z - z_2)(z - z_3)(z - z_4) \dots (z - z_n) \quad (227)$$

there follows

$$\prod_2(z) = (z - z_3)(z - z_4) \dots (z - z_n) \quad (228)$$

and therefore

$$\lambda_2 = 1 - \frac{1}{z_1^{n-2}} (z - z_3)(z - z_4) \dots (z_1 - z_n) \quad (229)$$

or

$$\lambda_2 = 1 - \left(1 - \frac{z_3}{z_1}\right) \left(1 - \frac{z_4}{z_1}\right) \dots \left(1 - \frac{z_n}{z_1}\right) \quad (230)$$

By interchanging the indices 1 and 2 for λ and z in Equations (222), (226) and (230), similar expressions are obtained for λ_1 .

VITA

VITA

André Georges Vacroux was born on May 22, 1936 in Paris, France. He is the son of Georges Vacroux and Eve Ter Akopoff-Vacroux. He is a citizen of France.

Mr. Vacroux received his primary and secondary education from the Collège Stanislas in Paris, France and received his Baccalauréats with honors, the first part in 1952 with a major in Humanities, Latin and Modern Languages and the second part in 1953 with a major in Mathematics.

Then Mr. Vacroux prepared the entrance examination and was admitted to the Ecole Supérieure d'Electricité in Malakoff, France from which he received the Diplôme d'Ingénieur Radio-E.S.E. in 1959.

In the meantime he attended also the Sorbonne (Mathématiques Générales) and the Ecole Nationale des Langues Orientales Vivantes where he majored in Russian Language and Culture.

Mr. Vacroux came to the U.S.A. in the fall of 1959 upon receiving a Teaching Fellowship from the University of Notre Dame and a Fulbright Grant.

While at the University of Notre Dame during the year 1959-1960, Mr. Vacroux was in charge of the Electronics Laboratory. A M.S.E.E. was granted to him in 1961, the topic

of his thesis being "A Method for Converting Continuous Voltages to Digital Signals".

Since the fall of 1960 Mr. Vacroux has been doing graduate work at Purdue University in the Department of Mechanical Engineering under the supervision of Professor Rufus Oldenburger. He is also employed in research work sponsored by the National Aeronautics and Space Administration of the Government of the United States at the Automatic Control Center. His doctoral dissertation is entitled "On Finding Quadratic Factors of Polynomials".

Upon completion of his degree, Mr. Vacroux is to return to France in order to fulfill his military obligations.

Mr. Vacroux is a member of the Société Française des Electroniciens et Radio-Electriciens and of the I.E.E.E.

Mr. Vacroux was employed as an engineer at the S.I.N.T.R.A. in Asnières, France during the summer of 1958. He has taught at Purdue University during the summers of 1961 and 1962.